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A Is for *Apple*: Mnemonic Symbols Hinder the Interpretation of Algebraic Expressions

Nicole M. McNeil
University of Notre Dame

Aaron Weinberg
Ithaca College

Shanta Hattikudur, Ana C. Stephens, Pamela Asquith, Eric J. Knuth, and Martha W. Alibali
University of Wisconsin–Madison

This study examined how literal symbols affect students' understanding of algebraic expressions. Middle school students ($N = 322$) were randomly assigned to 1 of 3 conditions in which they were asked to interpret an expression (e.g., $4c + 3b$) in a story problem. Each literal symbol represented the price of an item. In the *c*-and-*b* condition, the symbols used were the 1st letters of the items (e.g., price of a cake in dollars = *c*; price of a brownie in dollars = *b*). In the other 2 conditions, *c* and *b* were replaced with nonmnemonic English letters (*x* and *y*) or Greek letters (Φ and Ψ). Incorrect interpretations of the expression were most common among students in the *c*-and-*b* condition. Moreover, students in this condition were more likely than students in the other conditions to misinterpret the symbols as labels for objects (e.g., *c* stands for *cake*). An analysis of participating students' textbooks revealed that mnemonic symbols were used correctly and were not uncommon. Results suggest that the use of mnemonic symbols may hinder students' interpretation of algebraic expressions.

Keywords: mathematics, symbols, algebra, problem solving, cognitive development

Algebra is the foundation of higher order mathematics and science. It is so important, in fact, that students are required to pass a year of Algebra to graduate from high school in many districts. The National Council of Teachers of Mathematics (NCTM; 2000) has described some of the fundamental concepts that are prerequisites for success in algebra. These include understanding what a variable is, using variables in geometric formulas and linear equations, and understanding what algebraic symbols represent. Unfortunately, many students struggle to understand fundamental algebraic concepts (e.g., Bell, 1995; Booth, 1988; Herscovics & Linchevski, 1994; Kieran, 1992; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Küchemann, 1981; MacGregor & Stacey, 1997; Sfard, 1991; Sfard & Linchevski, 1994; Stephens, 2003, 2005). In particular, students seem to have considerable difficulties understanding and interpreting the symbolic notation used in algebra (e.g., Kinzel, 1999). In the present article, we focus on students' understanding of symbolic representations of variables.

Variables are typically denoted by literal symbols (e.g., *x*, *y*, *z*), and they serve a variety of roles in mathematics, including a means for expressing generalized arithmetic (e.g., $a + 0 = a$), a means for representing an "unknown" number (e.g., $5x + 3 = 18$), an argument of a function (e.g., $\sin(x)$), and a constant (e.g., $d = 1/2gt^2$, in this case *g*), among others (see Usiskin, 1988). Given the various manifestations of literal symbols, it may not be surprising that many students have misconceptions about what they represent. These misconceptions are revealed when students are asked to interpret algebraic expressions (Küchemann, 1978; Stephens, 2003), translate sentences into equations (Clement, Lochhead, & Monk, 1981; Fisher, 1988; Paige & Simon, 1966), or solve algebraic equations (Herscovics & Linchevski, 1994). Some of the most common misconceptions include assigning a value to a literal symbol that corresponds to its position in the alphabet (e.g., $a = 1$, $b = 2$, etc.; MacGregor & Stacey, 1997), viewing a literal symbol as a place holder for a missing digit (e.g., if $n = 5$, $3n = 35$; Herscovics & Chalouh, 1985), haphazardly assigning a specific value to a literal symbol (e.g., $3n$ is always bigger than $n + 6$ because 3×5 is 15 and $5 + 6$ is 11; Knuth et al., 2005), and interpreting a literal symbol as a shorthand label for an object or unit (e.g., *n* stands for *newspapers*; Booth, 1988; Knuth et al., 2005; Küchemann, 1978; MacGregor & Stacey, 1997; Paige & Simon, 1966; Rosnick, 1981).

This last interpretation is the focus of the current study. Rosnick (1982) documented students' difficulties with "semantically laden" symbols—letters that are often used in word problems and

Nicole M. McNeil, Department of Psychology, University of Notre Dame; Aaron Weinberg, Department of Mathematics, Ithaca College; Shanta Hattikudur and Martha W. Alibali, Department of Psychology, University of Wisconsin–Madison; Ana C. Stephens, Pamela Asquith, and Eric J. Knuth, Department of Curriculum and Instruction, University of Wisconsin–Madison.

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Correspondence concerning this article should be addressed to Nicole M. McNeil, Department of Psychology, University of Notre Dame, 118 Haggar Hall, Notre Dame, IN 46556. E-mail: nmcneil@nd.edu

stand "not only for numbers alone but for numbers or amounts of some qualitative entity to which they refer" (p. 2). For example, he interviewed one student about the equation $6R + 2B = 40$, given the following information: books cost $2 each, records cost $6 each, the same number of books and records were purchased, and $40 was spent in total. Over the course of the interview, the student simultaneously associated the symbol *B* with multiple attributes such as "books," "quantity of books," "price per book," and "total price of all the books." Nine of the 10 students interviewed exhibited similarly loose interpretations of literal symbols. Rosnick concluded, "Apparent shifts and contradictions are for the student only different manifestations of one broad, undifferentiated concept" (p. 19). These shifts and contradictions may be due, at least in part, to students' tendency to treat the literal symbols as labels for the entities being discussed in the problem (e.g., *B* stands for *books*). Indeed, Lucariello (in press) recently developed a computer-based multiple-choice test to help middle and high school teachers diagnose students' variable misconceptions. She tested over 450 students in Grades 6–12 (with the majority coming from Grades 8 and 9), and she found that the most prevalent variable misconception held by students is interpreting literal symbols as shorthand labels for objects (e.g., *s* stands for *students*).

Küchemann (1978) had previously uncovered this misconception in his seminal study of 13- to 15-year-old students' understanding of variables. Students in his study were presented with the following problem (among others): "Cakes cost *c* pence each and buns cost *b* pence each. If I buy 4 cakes and 3 buns, what does $4c + 3b$ stand for?" Only 22% of 14-year-old children answered the problem correctly (e.g., "the total cost of cakes and buns"), and 39% said that the expression $4c + 3b$ stood for "4 cakes and 3 buns." On the basis of his research, Küchemann (1978, 1981) concluded that students cannot "cope" with literal symbols as variables until they reach the highest substage of Piagetian formal operational thought (around age 15). He reasoned that before that point, when students are in the late substage of concrete operational thought or the early stage of formal operational thought, they tend to interpret literal symbols as shorthand labels for objects or as specific, fixed unknown numbers. This view is consistent with traditional accounts of students' difficulties with algebra, which attribute misconceptions to students' age or level of cognitive development (Filloy & Rojano, 1989; Halford, 1978; Herscovics & Linchevski, 1994; Sfard & Linchevski, 1994). However, more recent studies by Knuth et al. (2005) and MacGregor and Stacey (1997) suggest that age alone cannot account for students' misconceptions about variables (see also Paige & Simon, 1966).

In Knuth et al.'s (2005) recent study, 13-year-old students were asked what the symbol *n* stands for in the expression $2n + 3$. A majority (85%) interpreted the symbol as a variable, and few (less than 10%) interpreted it as a shorthand label for objects (see also Asquith, Stephens, Knuth, & Alibali, 2007). The authors speculated that this high level of performance could be attributed to the students' experience with Connected Mathematics, a standards-based curriculum that provides opportunities for students to work with literal symbols in ways that support the interpretation that a single literal symbol can stand for multiple values (i.e., a multiple-values interpretation). This speculation parallels the results of an influential study by MacGregor and Stacey (1997). They examined 11- to 15-year-old students' interpretations of variables and found that students' interpretations depended heavily on the types of teaching materials used in their classrooms. Specifically, they found that the symbols-as-labels misconception was prevalent only among students who had been exposed to teaching materials that presented literal symbols as abbreviated words (e.g., "*c* could stand for 'cat,' so 5*c* could mean 'five cats'"; p. 14). These students generally made poor progress in their understanding of literal symbols as variables over the course of 13 months. However, students who had not been exposed to such teaching materials, but instead encountered literal symbols as expressions of quantities, made good progress over the same period.

The findings of Knuth et al. (2005) and MacGregor and Stacey (1997) suggest that students under the age of 15 can interpret literal symbols as variables if they are exposed to curricular materials that support such a view. Unfortunately, what constitutes "appropriate support" is still an empirical question. In traditional approaches to algebra instruction, students often start by analyzing patterns in single variable data. They learn how to solve linear equations that have only one variable (i.e., an unknown, typically represented by *x*), and they receive extensive practice with manipulating terms and solving for unknown values in symbolic form. Materials based on this approach do not seem to promote conceptual understanding of literal symbols as variables. Indeed, as Schoenfeld (1988) argued, one of the "dramatic failures" of the traditional approach is that "students who are capable of performing symbolic operations in a classroom context . . . often fail to map the results of the symbolic operations they have performed to the systems that have been described symbolically" (p. 150).

More recent views of algebra instruction—views that consider algebra as a K–12 strand—advocate the need to provide students with informal problem-solving situations and other "early algebraic" experiences that are designed to foster intuitive understanding of generalized arithmetic (Blanton & Kaput, 2003, 2005; Carpenter, Franke, & Levi, 2003; Carraher, Schliemann, & Brizuela, 2006; NCTM, 2000). Within this approach, even very young students are frequently taught to use literal symbols as tools for expressing intuitive ideas about varying quantities in real-world problem situations (Carraher et al., 2006; Schoenfeld & Arcavi, 1988; Stephens, 2005). Third-grade students in Carraher et al.'s (2006) study, for example, were able to express the notion of $3 more than an unknown starting amount of money as $N + 3$, and were able to express values on a number line in relation to an arbitrary value. Thus, materials based on this approach seem to help children develop a conceptual understanding of variables (Blanton & Kaput, 2005; Brizuela & Schliemann, 2004; Carraher et al., 2006; Kaput, 2000).

In light of the success of these (and other) efforts to teach young students to use literal symbols properly, it is tempting for educators to give a blanket endorsement to any approach that makes mathematics less abstract and more connected to the real world (e.g., Burns, 1996; NCTM, 2000). One such approach is using mnemonic symbols to activate real-world knowledge of objects and quantities. The rationale seems clear: If children are better able to understand abstract mathematics concepts in contexts that appeal to their intuitive, real-world knowledge, then teachers should do everything they can to foster connections between school mathematics and the real world. However, it can be precarious to assume that a particular approach facilitates children's understanding simply because it appears to forge such connections. Without systematic experiments to tease apart the factors responsible for the success of well-designed curricula, educators are left to rely on intuition, and they may adopt approaches that are ineffective or detrimental.

For example, interviews with middle school math teachers (e.g., Asquith et al., 2007) have shown that some teachers intentionally steer away from traditional literal symbols (x and y) toward mnemonic literal symbols (e.g., the number of apples might be represented as a, and the number of pears might be represented as p). This approach, which has been referred to as the "fruit salad algebra" approach (Bell, 1995; Tall, 1993), often reflects an attempt on the part of a teacher to foster links between the abstract symbols and the real world. However, it is unclear whether this approach actually facilitates students' algebraic reasoning and, in particular, their understanding of literal symbols. Some researchers and educators suggest that it is useful because it helps students link abstract mathematical ideas to the real world or because it might help students understand the rules of simplification and collecting like terms (e.g., a teacher might say, "Let's say a is apples and b is bananas and suppose you've got $a + b + b + a + a$. Now apples and bananas are different so you have to count them up separately—that's three apples and two bananas, or $3a + 2b$"; Duke & Graham, 2007, p. 43). However, others point to the misconceptions documented by Küchemann (1978), Rosnick (1982), and Lucariello (in press) and warn that using mnemonic literal symbols may be detrimental if it strengthens students' naive conception that literal symbols in algebraic expressions stand for labels instead of quantities (see also Stacey & MacGregor, 1997).

Those who argue against the use of mnemonic literal symbols note that students arrive in algebra class having already used letters in many ways. In addition to using letters to write and interpret secret codes (e.g., *A* stands for 1, *B* stands for 2, etc.; Braddon, Hall, & Taylor, 1993), students commonly use letters as abbreviations (e.g., *N* stands for *north*, *c* stands for *cup*, *s* stands for *seconds*) and acronyms (e.g., USA), even in math class. Moreover, it would not be uncommon for a student or teacher to express the relationship "there are three feet in one yard" informally as "3 F = 1 Y" (Kaput & Sims-Knight, 1983, p. 71). Having already developed these ways of thinking about letters, students in algebra are expected to begin constructing new ways of thinking about letters that incorporate variables into their many uses.

As many researchers have shown, different ways of thinking about a particular concept can be activated depending on the context (Barsalou, 1982; McNeil & Alibali, 2005a; McNeil et al., 2006; Munakata, McClelland, Johnson, & Siegler, 1997; Strohner & Nelson, 1974). When a particular way of thinking about a concept is well established, as is the letters-as-labels interpretation, it is relatively easy to activate across a wide range of contexts. In contrast, when a particular way of thinking about a concept is newly emerging, as is the letters-as-variables interpretation, it may or may not be activated, depending on the context. According to this view, middle school students may need substantial contextual support to help them interpret letters as variables. Students' interpretation of letters as variables may be hindered in contexts that strengthen the letters-as-labels interpretation (e.g., use a to represent the number of apples) and helped in contexts that do not strengthen the letters-as-labels interpretation (e.g., use x to represent the number of apples).

In the present study, we examined whether the type of symbols used affects students' interpretations of algebraic expressions. We built on the seminal work of Küchemann (1978, 1981) by analyzing middle school students' interpretation of the algebraic expression in the cakes-and-buns problem. Recall that few students in Küchemann's (1978) study interpreted the expression $4c + 3b$ correctly. We hypothesized that students' poor performance could be attributed, at least in part, to the mnemonic nature of the symbols used in the problem (i.e., the price of a cake was represented as c, and the price of a bun was represented as b). When symbols are mnemonic, they may activate and support students' naive conception that literal symbols in algebraic expressions stand for labels instead of quantities. To complement our experiment, we also analyzed the types of symbols used in students' mathematics textbooks to see whether the students in our experiment had any formal exposure to mnemonic symbols. In the following sections, we report the main experiment followed by the textbook analysis.

Experiment

Method

Participants. Participants were 110 sixth-grade students (44 boys, 66 girls), 119 seventh-grade students (57 boys, 62 girls), and 93 eighth-grade students (48 boys, 45 girls) recruited from a public middle school in the midwestern United States. The racial or ethnic makeup of the school was 24% African American, 7% Asian, 6% Hispanic, and 63% White. Approximately 37% of students received free or reduced-price lunch. Approximately 70%–80% of students in each grade level at the school scored at or above "proficient" on the math section of the yearly state standardized test of academic progress, which was similar to the average for students at middle schools across the state. The school used a standards-based math curriculum, Connected Mathematics, which it had adopted a few years prior to this study. The sixth-grade teachers were in their fourth year of implementation, and the seventh- and eighth-grade teachers were in their fifth year of implementation. Thus, all students in this study had been exposed to Connected Mathematics for their entire middle school career. Connected Mathematics develops children's algebraic reasoning starting in the sixth-grade units, with increasing attention given to algebra in the seventh- and eighth-grade units. Topics traditionally associated with algebra such as solving linear equations receive explicit attention beginning in the seventh-grade units.

Procedure. We presented students with the following question, adapted from Küchemann (1978, 1981): "Cakes cost c dollars each and brownies cost b dollars each. Suppose I buy 4 cakes and 3 brownies. What does $4c + 3b$ stand for?" We embedded this question in a paper-and-pencil test designed to assess students' understanding of algebraic concepts and procedures. Students were randomly assigned to one of three conditions. The only difference across conditions was the specific symbols used in place of c and b in the question. In the c-and-b condition, the price of a cake was represented by c, and the price of a brownie was represented by b (as above). This mnemonic condition was a replication of Küchemann's (1978, 1981). In the x-and-y condition, c and b were replaced by the traditional, nonmnemonic letters x and y. In the Φ-and-Ψ condition, c and b were replaced by Φ and Ψ, which are neither mnemonic nor familiar to middle school students.

Coding. Students' responses were first coded according to whether they were correct or incorrect. A student was given credit for a correct response if he or she indicated that the letters stood for the cost or price of the cakes and brownies (e.g., "four times the price of a cake plus three times the price of a brownie"). Correct responses were further coded according to whether students exhibited a structural or operational interpretation of the algebraic expression (Sfard, 1991; see also Dubinsky, 1992; Sfard & Linchevski, 1994). In a structural interpretation, the symbols are treated as a single objectlike entity (e.g., $3n$ would be interpreted as a single quantity). An example of a structural interpretation of the expression $4c + 3b$ is "the total cost of four cakes and three brownies." In contrast, in an operational interpretation, the symbols are viewed in terms of processes or procedures (e.g., $3n$ would be interpreted as $3 \times n = \_\_$). An example of an operational interpretation of the expression $4c + 3b$ is "four times c plus three times b." Both are important components of mathematical thinking, but viewing mathematical ideas structurally is thought to be a prerequisite for understanding more advanced mathematical concepts (Gray & Tall, 1994; Sfard, 1991; Sfard & Linchevski, 1994). For example, consider a student who views an algebraic expression such as $x + 2$ operationally (i.e., as the process of taking some number and adding 2) but cannot view it structurally (i.e., as the result of adding 2 to some number). Such a student would have difficulty determining the perimeter of a rectangle with two sides of length x and two sides of length $x + 2$ because the student would not be able to operate meaningfully on the quantity $x + 2$ to obtain $4x + 4$ as the final solution.

Incorrect responses were further coded according to whether students interpreted the literal symbols as labels for the objects being depicted in the problem rather than for the price of the objects (e.g., "it stands for four cakes and three brownies"). Examples of students' responses along with their respective codes are presented in Table 1.

Reliability for coding students' responses was established by having a second coder evaluate the interpretations of a randomly selected 20% sample. Agreement between coders for coding whether students were correct was 94%. Agreement between coders for coding whether students exhibited a structural interpretation of the expression was 94%. Agreement between coders for coding whether students interpreted the literal symbols as labels for the objects was 98%.

Results and Discussion

We used logistic regression to examine the association between the type of symbol used in the algebraic expression and the likelihood of interpreting the algebraic expression correctly. Predictor variables included symbol condition (c and b, x and y, or Φ and Ψ) and grade level (6–8). Two Helmert contrast codes were used to represent the two degrees of freedom of symbol condition: (a) c and b versus the two nonmnemonic conditions and (b) x and y versus Φ and Ψ.

Figure 1 displays the percentage of students in each grade level who interpreted the algebraic expression correctly in each symbol condition. As shown in the figure, students in the c-and-b condition were less likely than those in the nonmnemonic conditions to interpret the algebraic expression correctly (42 of 112 [37%] vs. 117 of 210 [56%]), $B = -0.76$, $z = -2.96$, $Wald\ (1, N = 322) = 8.81$, $p < .01$. The model estimates that the odds of interpreting the algebraic expression correctly are more than two times lower in the c-and-b condition than in the nonmnemonic conditions. Students in the x-and-y condition and students in the Φ-and-Ψ condition were equally likely to interpret the algebraic expression correctly (60 of 105 [57%] vs. 57 of 105 [54%]), $B = 0.12$, $z = 0.41$, $Wald\ (1, N = 322) = 0.17$, $p = .68$. The likelihood of interpreting the algebraic expression correctly also increased with grade level, $B = 0.97$, $z = 6.09$, $Wald\ (1, N = 322) = 37.05$, $p < .01$.

The results are similar when we consider whether students exhibited a structural interpretation of the algebraic expression (e.g., "the total cost of 4 cakes and 3 brownies"). Figure 2 displays the percentage of students in each grade level who exhibited a structural interpretation of the algebraic expression in each condition. As shown in the figure, students in the c-and-b condition were less likely than those in the nonmnemonic conditions to exhibit a structural interpretation (12 of 112 [11%] vs. 45 of 210 [21%]), $B = -0.80$, $z = -2.24$, $Wald\ (1, N = 322) = 5.03$, $p = .02$. Students in the x-and-y condition and students in the Φ-and-Ψ condition were equally likely to exhibit a structural interpretation (23 of 105 [22%] vs. 22 of 105 [21%]), $B = 0.051$, $z = 0.15$, $Wald\ (1, N = 322) = 0.022$, $p = .88$. Consistent with Sfard's (1991) hypothesis that the structural interpretation is the most mathematically advanced interpretation, the likelihood of exhibiting a structural interpretation of the algebraic expression also increased with grade level, $B = 0.78$, $z = 3.86$, $Wald\ (1, N = 322) = 14.87$, $p < .01$.

Recall that we predicted that students' performance would be poorest in the c-and-b condition because the symbols in that condition are mnemonic, and they overlap with students' knowledge of letters as labels. Figure 3 displays the percentage of students in each grade level who misinterpreted the algebraic expression in this way (e.g., "4 cakes and 3 brownies"). As shown in the figure, students in the c-and-b condition were more likely

Table 1
Examples of Students' Interpretations of the Algebraic Expression 4c + 3b

Interpretation	Primary code	Secondary code
It stands for 4 cakes plus 3 brownies	Incorrect	As labels
Four cakes and three brownies	Incorrect	As labels
It could be anything	Incorrect	Not as labels
I think it stands for $4x + 3y = ?$	Incorrect	Not as labels
Four times the cost of a cake plus three times the cost of a brownie	Correct	Operational
(3 brownies times the cost of 1 brownie) + (4 cakes times the cost of 1 cake)	Correct	Operational
The amount of money it would cost to buy 4 cakes and 3 brownies	Correct	Structural
The total amount of money you spent	Correct	Structural

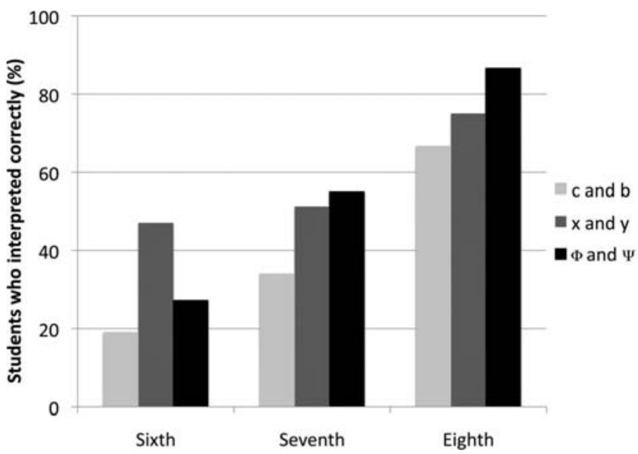

Figure 1. Percentage of children in each condition who interpreted the algebraic expression correctly.

than those in the nonmnemonic conditions to misinterpret the symbols as labels for the objects (30 of 112 [27%] vs. 17 of 210 [8%]), $B = 1.48$, $z = 4.35$, *Wald* $(1, N = 322) = 18.91$, $p < .01$. Students in the *x*-and-*y* condition and students in the Φ-and-Ψ condition were equally unlikely to misinterpret the expression in this way (11 of 105 [10%] vs. 6 of 105 [6%]), $B = 0.66$, $z = 1.24$, *Wald* $(1, N = 322) = 1.55$, $p = .21$. Thus, as predicted, students in the mnemonic condition were more likely than students in the nonmnemonic conditions to interpret the literal symbols as labels. However, it is important to point out that some students in the nonmnemonic conditions did interpret the symbols as labels (e.g., they said that $4x + 3y$ stood for "4 cakes and 3 brownies"), so the use of nonmnemonic symbols did not completely eliminate this interpretation. The likelihood of misinterpreting the symbols as labels was not associated with grade level, $B = 0.11$, $z = 4.07$, *Wald* $(1, N = 322) = 0.29$, $p = .59$.

Although the symbols-as-labels error did not decrease with students' grade level, we know that some types of errors must have decreased because correct responses increased with grade level (see previous results). To examine this issue more closely, we examined the other types of errors students made. Apart from interpreting the symbols as labels (29% of all errors), students made other errors that fell into one of four categories. First, students sometimes gave no response or responded by indicating an inability to answer the problem ("I don't know," "not sure," "no idea"). These errors accounted for 38% of all errors. Second, students sometimes responded in vague, incomplete, or uninterpretable ways (e.g., "my guess is 410 + 33," "4 dollars and 3 dollars cakes cost $1.00 brownies cost $1.00," "how much stuff you get," "y *x*'s and x *x*'s," "*x* stands for dollar sign & *y* stands for cents"). These errors accounted for 19% of all errors. Third, students sometimes restated the expression without providing any additional information about the meaning of the variables (e.g., "$(4 \times x) + (3 \times y)$," "it stands for $4 \times c$ and $3 \times b$," "you times $4 \times \Phi$ then you times $3 \times \Psi$ then you add those answers together"). These errors accounted for 9% of all errors. Finally, students sometimes incorrectly added the unlike terms to get 7 (e.g., "$7," "7*xy*," "it stands for seven of them"). These errors accounted for 5% of all errors. The first three of these error types decreased with grade level as follows: Giving no response or responding by indicating an inability to answer the problem accounted for 35% of all responses in sixth grade, 17% in seventh grade, and 3% in eighth grade. Responding in vague, incomplete, or uninterpretable ways accounted for 12% of all responses in sixth grade, 10% in seventh grade, and 6% in eighth grade. Simply restating the expression without providing any additional information about the meaning of the variables accounted for 8% of all responses in sixth grade, 4% in seventh grade, and 1% in eighth grade. The final error type (adding unlike terms) accounted for 4% in sixth grade, 2% in seventh grade, and 2% in eighth grade.

The results of our experiment suggest that students are most likely to struggle with the interpretation of expressions involving literal symbols when those symbols are mnemonic ones. They are less likely to answer correctly, less likely to provide structural interpretations, and more likely to interpret the symbols as labels. Might this be because these students have little experience with mnemonic symbols, or because they are exposed to materials that present mnemonic literal symbols incorrectly as abbreviated words (as in MacGregor & Stacey's, 1997, School C)? To find out, we undertook an analysis of the middle school textbook series used by the students in our experiment. We chose to examine the textbooks because they are likely to influence teachers' beliefs, teachers' instructional practices, and students' learning (Nathan, Long, & Alibali, 2002).

Textbook Analysis

Method

Materials. We examined the textbook series used by the middle school students in the experiment—Connected Mathematics (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). This textbook series is a curriculum whose creation was supported by National Science Foundation funding and whose design was guided by NCTM's *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). It emphasizes conceptual understanding with a special emphasis on using concrete examples and connecting mathematics to the real world.

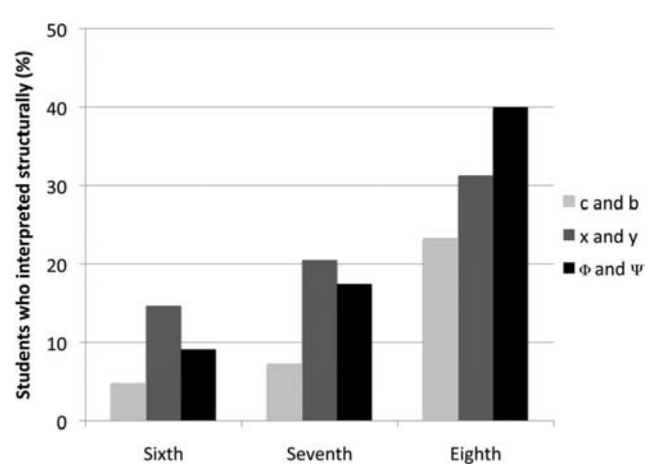

Figure 2. Percentage of children in each condition who interpreted the algebraic expression structurally.

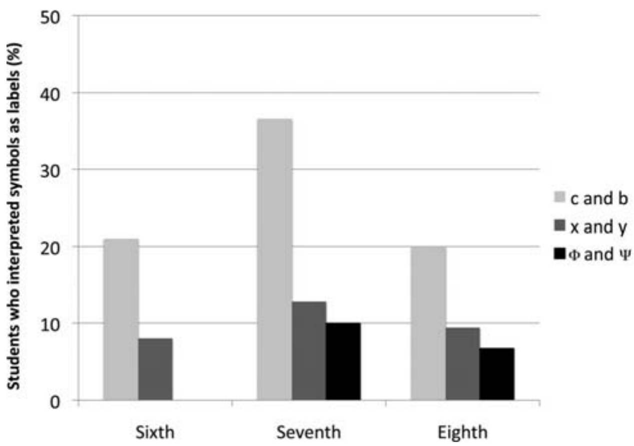

Figure 3. Percentage of children in each condition who interpreted the literal symbols as labels for objects.

Coding. In line with the method used by McNeil et al. (2006), we looked at a randomly selected 50% sample of the pages in every book (Grades 6–8). We coded every instance of a variable that was represented by a symbol. Each instance was first coded according to whether the symbol was a question mark (e.g., $-17 + 13 = ?$), a letter in the English alphabet (e.g., $y = 6$), or a letter in the Greek alphabet (e.g., π). Then, the English letter code was further divided into one of two subcategories: (a) mnemonic, in which the letter corresponded to the first letter of the word for an object, unit, substance, quantity, idea, etc., being represented in the problem (e.g., number of ancestors, *A*, in a given generation) or (b) nonmnemonic, in which the letter did not correspond to the first letter of the word for an object, unit, substance, quantity, idea, etc., being represented (e.g., let *x* represent the distance in meters). Reliability was established by having a second coder evaluate a randomly selected 20% of the sample. Agreement between coders was 100% for coding whether a symbol was a question mark, English letter, or Greek letter, and 100% for coding whether a letter was mnemonic or nonmnemonic.

Results and Discussion

Table 2 presents the total number of symbols found in each grade level's 50% sample, along with the number of pages in the 50% sample and the average number of symbols per page. It is not surprising that the number of symbols increased from sixth to eighth grade. This is true whether one considers the total number of symbols or the average number per page.

Table 2 displays the proportion of symbols that were question marks, English letters, or Greek letters. As shown in the table, English letters were found more often than question marks or Greek letters across all grade levels. Question marks were more likely to be found in sixth grade than in seventh or eighth grade.

Table 2 displays the proportion of English letters that were mnemonic (vs. nonmnemonic). As shown in the table, English letters were less likely to be mnemonic than nonmnemonic. However, the use of mnemonic letters was not rare. Indeed, mnemonic letters accounted for nearly a third (28%) of the English letter symbols seen by students across the three grade levels. Thus, students did have quite a bit of experience with letters being used mnemonically in their textbooks.

When mnemonic letters were used in the textbook, they were derived from the real-world problem context in one of four ways. The first, most common way (56% of the instances) was to use the first letter of the physical quantity being measured (e.g., "write an equation for the area, *A*," "the relationship between a person's height, *H*," "*d* represents the distance traveled in miles"). The second way (11% of the instances) was to use the first letter of the units being used to measure the physical quantity (e.g., "predict the miles driven (*m*)," "if the cost for car insurance is *d* dollars per year"). The third way (10% of the instances) was to use the letter *n* to denote some *number* of objects (e.g., "*n* is the number of T-shirts sold," "and the number of containers (*n*)"). The fourth way (23% of the instances) included all remaining mnemonic instances in which the literal symbol was the first letter of one of the objects, entities, or substances involved in the problem (e.g., "write an equation for the number of handshakes, *h*," "*t* is the number of spare tires needed for *c* customers," "predict the number of visitors based on the probability of rain, *R*"). To highlight the difference between these three categories, consider a problem involving a car that has traveled some number of miles between Point A and Point B. A textbook could use *d* to represent the distance traveled in miles (physical quantity being measured), *m* to represent the distance traveled in miles (units being used to measure the physical quantity), *n* to represent the number of miles traveled (number), or *c* to represent the distance traveled by the car in miles (first letter of object doing the traveling). It is important to note that even though mnemonic letters were used in various ways, they were never used incorrectly. We did not encounter a single instance in the Connected Mathematics textbook series in which mnemonic literal symbols were presented in a misleading way as abbreviated words (e.g., "*c* could stand for 'cat,' so 5*c* could mean 'five cats'"; MacGregor & Stacey, 1997, p. 14).

Thus, on the basis of our analysis of students' textbooks, we do not suspect that students' poor performance in the mnemonic condition was due to a lack of experience with mnemonic symbols. The fact that we found very few instances of Greek letters being used as symbols (1% across grade levels, and no instances of Φ or Ψ) along with our finding that students performed equally well in the *x*-and-*y* and Φ-and-Ψ conditions strengthens the argument that simple exposure to particular types of symbols did not influence students' performance in our experiment. Furthermore, our textbook analysis suggests that students' poor performance in the

Table 2
Textbook Analysis of Symbolic Instances Across Grades 6–8

Sample	Grade 6	Grade 7	Grade 8
Number of pages	314	327	285
Quantity of instances			
Total	70	662	1,606
Instances per page	0.22	2.02	5.64
Proportion of instances			
Question marks	.44	.04	<.01
English letters	.53	.95	.99
Greek letters	.03	.01	.01
Proportion of English letters that were mnemonic	.15	.32	.27

mnemonic condition cannot be attributed to explicit exposure to these types of symbols being used in misleading ways in the textbook. However, it remains possible that any exposure to mnemonic symbols—even exposure to mnemonic symbols that are used correctly—may actually reinforce students' incorrect interpretations of the symbols as labels. Because students are not routinely questioned about the literal symbols they encounter in their textbooks, it is hard to know what knowledge is being activated and reinforced by the exposure.

General Discussion

In the present study, students were asked to interpret an algebraic expression that stood for the total cost of four cakes and three brownies (e.g., $4c + 3b$), and their performance depended on the symbols used in the expression. As predicted, performance was better when the symbols used in the expression were not mnemonically related to the objects (e.g., x or Φ was used to represent the price of a cake in dollars) than when the symbols were mnemonically related to the objects (e.g., c was used to represent the price of a cake in dollars). This result held regardless of whether successful performance was operationalized as presence of a mathematically correct interpretation, presence of a structural interpretation, or absence of a symbols-as-labels interpretation. Moreover, an analysis of students' mathematics textbooks revealed that students' difficulties with mnemonic literal symbols might not be due to their being exposed to incorrect or misleading examples in their textbooks. On the contrary, when mnemonic literal symbols were presented in students' textbooks, they were used correctly to represent quantities. Taken together, these results suggest that mnemonic literal symbols may not constitute the appropriate contextual support that middle school students need to help them interpret letters as variables. Thus, the poor performance of middle school students on the cakes-and-buns problem in Küchemann's (1978) classic study was likely due, at least in part, to the mnemonic nature of the symbols used in the problem, rather than solely to limitations associated with students' age or general level of cognitive development.

Although the results of the present study supported our predictions, we cannot state with certainty the reasons for students' poor performance with mnemonic literal symbols. There are at least three possible explanations. First, students' prior experience with letters may hinder their ability to interpret mnemonic literal symbols as variables. Starting in preschool, children learn to associate the first letter of a word with the whole word (e.g., *A* is for *Apple*, *C* is for *Cookie*). Children continue to see letters used as shorthand labels throughout elementary school, even in math class (e.g., *g* for *gram*, *m* for *meter*). Thus, the letters-as-labels interpretation is fairly well established before students start learning about letters as variables in middle school. In contrast, the letters-as-variables interpretation is not well established before middle school, so it may need appropriate contextual support to be strongly activated and selected (cf. Barsalou, 1982; McNeil & Alibali, 2005a). When children have multiple interpretations that compete with one another for precedence, the environment can play a large role in which interpretation is selected (Siegler, 1996). In the present study, we found that the mnemonic literal symbols were more likely to elicit the letters-as-labels interpretation and less likely to elicit the letters-as-variables interpretation in comparison to nonmnemonic literal symbols.

Second, the use of mnemonic literal symbols may draw children's attention to superficial aspects of the problem and away from the relevant mathematical concepts. One challenge in algebra is for students to think about more general properties of the real number system apart from characteristics of particular concrete referents. In the case of the cakes-and-brownies problem, students not only need to construct contextually appropriate interpretations of the act of purchasing cakes and brownies but also need to think about the more general algebraic concepts that generalize beyond the act of purchasing cakes and brownies, and even beyond the purchasing context. Several lines of research and theory suggest that generalization may be harmed when problem solvers attend to the superficial aspects of a problem (Bassok & Holyoak, 1989; Goldstone & Sakamoto, 2003; Kaminski, Sloutsky, & Heckler, 2008; McNeil, Uttal, Jarvin, & Sternberg, 2009; Uttal, Scudder, & DeLoache, 1997). In the present study, the use of the mnemonic symbols c and b may have cued students to think about the specific objects being purchased (i.e., cakes and brownies), and this increased attention to the superficial aspects of the problem may have distracted the students from the relevant concepts.

Third, students may be exposed to sources that explicitly present literal symbols in mathematics in misleading or incorrect ways. Despite the fact that students' middle school textbooks did not present mnemonic symbols in misleading or incorrect ways, there is no way of knowing whether other sources (e.g., elementary school textbooks, teachers, parents) may have done so. For example, how often do teachers refer to $4a$ as "4 apples" rather than "4 times the number of apples?" Similarly, how often do teachers help students learn to simplify algebraic expressions such as $3a + 2a$ by providing them with the following rationale: "I have 3 apples and 2 more apples, so I have 5 apples altogether"? It is, after all, teachers who govern students' use of textbooks, and it is teachers who need to take advantage of opportunities to solicit students' views about literal symbols and what they represent, so it is important to understand how teachers apply their own understandings of literal symbols in their teaching. Although we do not have quantitative data to speak to this issue, Asquith et al.'s (2007) interviews with students' math teachers suggest that there may be some cause for concern. For example, consider the following excerpt from one of the interviews with a sixth-grade teacher. The teacher was asked to consider the following problem: "Which is larger, $3n$ or $n + 6$?"

> *Interviewer:* How will students solve it?
>
> *Teacher:* They would do one of two things. One they'd put in numbers and test out numbers and see what works. The other thing I've taught them to substitute words in here, so they can understand what they're getting, which helps with exponents in particular. So like when it's $2a$ versus a^2. a^2 is a real big apple. $2a$ means you have two apples. So you know when you start to combine terms, two apples and a huge apple does not give you 4 apples. I use that real concrete. It's funny. It can get real goofy.
>
> *Interviewer:* When you make these objects, do you use the first letter?
>
> *Teacher:* Yes. Always. So we have lots of xylophones, x-rays cause x is like I wish there were more words for that. You know, we use lots of yo-yos. Giant sized yo-yos are very popular. Could they have

> picked better variables for us? So, I can see my kids going "you have one nut and you add six more to them and you have three times your one nut . . . three groups of them." So that would be one strategy I can surely see them using.
>
> *Interviewer:* To get them correct?
>
> *Teacher:* Yeah. Right.

As this excerpt illustrates, there is a need for further investigations of students' actual experiences with literal symbols.

Regardless of students' actual experience with incorrect uses of mnemonic literal symbols, the textbook analysis suggests that mere exposure to correct uses of mnemonic and nonmnemonic literal symbols in mathematics may not be sufficient on its own to dissuade the symbols-as-labels misinterpretation. As students progress from sixth to eighth grade in the Connected Mathematics curriculum, their exposure to mathematically correct uses of mnemonic and nonmnemonic literal symbols increases. For this reason, it is not surprising that correct interpretations and structural interpretations of the algebraic expression became more common as grade level increased. It was surprising, however, that use of the letters-as-labels interpretation did not become less common across the grade levels. This finding adds to a growing body of work suggesting that well-established percepts, concepts, and strategies persist even if they are inefficient or incorrect (e.g., Luchins, 1942; McNeil & Alibali, 2005b; Schauble, 1990; Siegler & Stern, 1998). When students develop a correct understanding of a concept, the new knowledge does not simply replace or subsume the old, incorrect knowledge. Instead, old knowledge structures continue to co-exist alongside the new ones, and these old ways of thinking can be activated, depending on the context (McNeil & Alibali, 2005a, 2005b; cf. Barsalou, 1982). Even undergraduates, who have years of experience with algebra and who presumably have reached the highest substage of formal operational thought, sometimes misinterpret mnemonic literal symbols as labels for objects (Clement, 1982). These results demonstrate the practical importance of students' early experiences in a domain.

Although our findings suggest that mnemonic symbols can hinder students' interpretation of algebraic expressions, two caveats should be noted. First, the mnemonic symbols in the present experiment, *c* and *b*, were derived from the first letters of the objects, cakes and brownies. It is possible that other types of mnemonic symbols (first letter of the physical quantity being represented, first letter of the units being used to measure the quantity, etc.) would not have been as detrimental to students' interpretation of the algebraic expression. For example, students may have been more successful if the *c*ost of a *c*ake had been represented by c_c and the *c*ost of a *b*rownie had been represented by c_b. These types of mnemonic symbols that are derived from the quantities being represented may actually facilitate students' interpretation of the problem at hand. However, even if we found these types of mnemonic symbols to be beneficial in the short term, we still might discourage educators from using them during instruction because mnemonic symbols—regardless of how they are derived—may reinforce students' naive conception that letters stand for labels (instead of for variables). We argue that it is detrimental to reinforce the symbols-as-labels conception in the long run because it may lead to errors when the symbols used in a problem have not been derived from the quantities being represented.

Second, the present experiment focused on students' interpretations of an algebraic expression that was written by a "knowledgeable other" (e.g., teacher, researcher, textbook writer). Findings may not generalize to situations in which students generate their own symbols and write their own algebraic expressions. Indeed, researchers have shown that students' self-generated ways of representing and solving math problems can sometimes lead to very different patterns of performance than those given to students by knowledgeable others (e.g., Nathan & Koedinger, 2000). These researchers recommend identifying students' intuitive ideas and bridging from those intuitive ideas when teaching new concepts. For example, an educator might introduce students to a word problem involving an unknown quantity (e.g., the cost of a brownie) and ask them to generate a representation of the given problem on paper using symbols (cf. Carraher et al., 2006). The process of generating representations may help students learn to use symbols meaningfully to represent unknown quantities. However, it is also possible that there could be a trade-off between helping students use symbols meaningfully and reinforcing the letters-as-labels conception, particularly if most students tend to generate symbols that are derived from the objects in the problem (e.g., use *b* to stand for the cost of a brownie). With this potential trade-off in mind, educators may be able to design classroom activities that capitalize on the potential strengths of having students generate their own representations while limiting the potential negatives of reinforcing the letters-as-labels conception. For example, after asking students to generate their own symbolic representations, an educator could encourage students to share their representations with the class and could lead the class in a discussion of the pros and cons of using different symbolic representations. Encouraging students to compare different strategies is one method that has been found to improve students' conceptual knowledge of algebra (Rittle-Johnson & Star, 2009), but benefits may depend on students' level of prior knowledge (Rittle-Johnson, Star, & Durkin, 2009). Without additional empirical evidence, the costs and benefits of using this approach to help students understand variables remain unclear.

Overall, the present study highlights the importance of analyses of curricula and teaching materials, not only in terms of the scope and sequence of topics but also in terms of the particulars of problem contexts and formats. We contend that it is important for educators to pay attention to the contexts and formats in which they present problems because small variations in how problems are presented can influence what students come to understand about the associated concepts. This has been shown not only for students' understanding of variables (as in the present study) but also for students' understanding of the equal sign (Baroody & Ginsburg, 1983; McNeil & Alibali, 2005a; McNeil et al., 2006; Seo & Ginsburg, 2003) and for simple algebra problems involving two operations and one unknown (Koedinger & Nathan, 2004). By studying how specific contextual manipulations affect students' performance, researchers can not only help educators choose materials for their classrooms, but also gain essential insights into how the environment affects the construction and organization of human knowledge.

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