Journal of Experimental Child Psychology 107 (2010) 15–30

Contents lists available at ScienceDirect

Journal of Experimental Child Psychology

journal homepage: www.elsevier.com/locate/jecp

Learning about the equal sign: Does comparing with inequality symbols help?

Shanta Hattikudur *, Martha W. Alibali

Department of Psychology, University of Wisconsin–Madison, Madison, WI 53706, USA

ARTICLE INFO

Article history:
Received 5 May 2009
Revised 17 March 2010
Available online 21 May 2010

Keywords:
Math learning
Comparison
Elementary school
Equal sign
Math education
Conceptual development

ABSTRACT

This study investigated whether instruction that involves comparing the equal sign with other relational symbols is more effective at imparting a relational interpretation of the equal sign than instruction about the equal sign alone. Third- and fourth-grade students in a comparing symbols group learned about the greater than, less than, and equal signs and had the opportunity to compare the inequality symbols with the equal sign. Students in an equal sign group learned about the equal sign only. A third group of students served as a control group. Three aspects of students' knowledge were assessed before and after the lesson: (a) conceptual understanding of the equal sign, (b) equation encoding, and (c) problem solving. Students in the comparing symbols group showed greater gains in conceptual understanding from pretest to posttest than students in the other two groups, and students in the comparing symbols group also scored higher on a posttest that assessed knowledge about inequality symbols and inequality problem solving. Thus, they learned about three symbols in the same amount of time as other students learned about the equal sign alone or not at all. Therefore, an instructional approach involving comparison can be an effective tool for learning about concepts in mathematics.

Introduction

Students use the equal sign extensively in arithmetic, but many students hold misconceptions about it. The purpose of the current study was to test an approach to instruction about the equal sign that involves comparison. Specifically, we investigated whether instruction that involves comparing

* Corresponding author.
E-mail address: hattikudur@wisc.edu (S. Hattikudur).

0022-0965/$ - see front matter
doi:10.1016/j.jecp.2010.03.004

the equal sign with other relational symbols is more effective at promoting a sophisticated relational interpretation of the equal sign than instruction about the equal sign alone.

Understanding of the equal sign in elementary and middle school

Many elementary and middle school students hold an *operational* view of the equal sign, interpreting it as meaning "the total" or "put the answer" (Baroody & Ginsburg, 1983; Kieran, 1981; Rittle-Johnson & Alibali, 1999). They view it as signaling a process or an action rather than a relationship between quantities (Kieran, 1992). In contrast, students who view the equal sign as expressing a relationship between quantities hold a *relational* view, interpreting the equal sign as meaning "the same as". Although an operational understanding of the equal sign may be sufficient for solving standard equations such as $3 + 4 = \_\_$, it can lead to difficulties with more sophisticated problems. For example, when solving equivalence problems such as $4 + 3 + 5 = 4 + \_\_$, many late elementary school students apply their operational understanding of the equal sign and add the numbers before the equal sign to get a solution of 12 or add all of the numbers to get 16 (e.g., Perry, Church, & Goldin-Meadow, 1988).

There also appear to be broader consequences to having an inadequate understanding of the equal sign. Students who hold an operational understanding of the equal sign have difficulties in the transition to algebra. More specifically, students who have an operational understanding perform more poorly than students who have a relational understanding in solving linear equations, even after controlling for mathematical ability (Knuth, Stephens, McNeil, & Alibali, 2006). Moreover, students who have an operational understanding learn less from lessons about how to solve linear equations (Grobman & Alibali, 2007).

Unfortunately, standard classroom instructional approaches used in the United States are often unsuccessful in helping students to develop a relational conception of the equal sign. Students' experiences with arithmetic may promote an operational understanding (Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980; Carpenter, Franke, & Levi, 2003; McNeil, 2007; Seo & Ginsburg, 2003). For example, an analysis of middle school curricular materials revealed that the equal sign is most often presented in an "operations equals answer" context (e.g., $3 + 4 = 7$), which is compatible with an operational interpretation (McNeil et al., 2006). Moreover, students who are asked to define the equal sign in this standard "operation equals answer" context are far less likely to offer a relational interpretation than students who are asked to define the symbol in a nonstandard context (e.g., $7 = 7$, $7 = 3 + 4$, $3 + 4 = 2 + 5$) (McNeil & Alibali, 2005a; McNeil et al., 2006). Students' experiences with arithmetic in "operations equals answer" contexts, therefore, may foster an incorrect view of the equal sign as an operational symbol rather than a relational one.

Another possible source of children's operational understanding may be the instruction they receive. A comprehensive study of six American preservice mathematics teacher textbooks revealed that little explanation was provided to teachers about how to teach students about the equal sign (Li, Ding, Capraro, & Capraro, 2008). One textbook even recommended that teachers use objects to show students that the equal sign means *how many objects there are altogether*—a suggestion compatible with an operational interpretation. Nevertheless, despite little or no direct instruction typically being provided to students about the equal sign's meaning, students become familiar with the symbol in equations and begin to make inferences about its meaning (e.g., Baroody & Ginsburg, 1983; Kieran, 1981; Rittle-Johnson & Alibali, 1999).

The purpose of the current study was to test an approach to instruction about the equal sign that involves *comparison.* Specifically, we investigated whether students gain a deeper relational understanding of the equal sign if they are invited to compare the equal sign with other relational symbols with which they are already familiar, specifically inequality symbols. We anticipated that comparison would foster in students a relational understanding of the equal sign. We based this prediction on past research from cognitive and educational psychology that focuses on the effects of comparison on learning (e.g., Gentner & Markman, 1994; Gentner & Medina, 1998; Schwartz & Bransford, 1998).

Effects of comparison on learning

A number of studies have documented that comparison can promote a deeper understanding of the concepts being compared. Gentner and colleagues have studied comparison as a means to facilitate an understanding of relational commonality or similarities between stimuli in regard to the relationships they share (e.g., father–son paired with mother–daughter, large–medium paired with medium–small) (Gentner & Markman, 1994; Gentner & Medina, 1998). When asked to choose between a relational match and nonrelational match to a target pattern, 4-year-olds were able to compare the two choices and select the pattern with higher order relational similarity over the pattern with nonrelational object similarity (Kotovsky & Gentner, 1996). Through comparison, children learned to recognize more abstract commonalities such as relational similarity.

Building on this past research about comparison, Schwartz and Bransford (1998) designed an instructional approach called *contrasting cases* instruction. Before teaching a lesson on the concept of *schema,* they divided a college lecture class into two groups: one that received six different accounts of a typical dentist's office visit and one that received only a one-page summary of the same accounts. Both groups were simply told to read the handouts before attending the following lecture. Students who compared and contrasted the six accounts scored higher than their classmates on subsequent multiple-choice and essay tests about schemas. Through detecting similarities and differences among several examples of the concept, students acquired a deeper understanding of the concept.

Rittle-Johnson and Star (2007) were the first to apply a contrasting cases approach in the domain of mathematics. They asked seventh-grade students to compare and contrast two solution procedures for solving the same algebra problem and found that these students were more likely to correctly solve posttest and transfer problems than other students who viewed the two methods sequentially. Engaging in conversation about the similarities and differences between the two procedures for solving the same problem was more beneficial than discussing the two procedures separately (Rittle-Johnson & Star, 2007; Star & Rittle-Johnson, 2009). Along similar lines, we believe that comparison can promote learning not only about procedures but also about concepts. Specifically, we predict that comparing the equal sign with a related, previously learned concept (inequality as expressed in inequality symbols) may allow students to notice important similarities while highlighting important distinctions.

Students are typically introduced to and become familiarized with comparing numbers and quantities in first grade, and they are taught to use inequality symbols shortly thereafter, as is evident in elementary school textbooks (e.g., Greenes et al., 2005; Kliman, Tierney, Russell, Murray, & Akers, 1997). The greater than and less than signs by definition require students to understand the relation between the quantities represented on the two sides of a mathematical statement. Comparing inequality symbols and the equal sign might allow students to connect their prior relational understanding of inequality symbols to the desired relational understanding of the equal sign and to view the equal sign as meaning *not greater than, not less than, but the same as.* Thus, we anticipate that comparison will lead to a deeper conceptual understanding of the equal sign.

The current study

In this study, we compared students' learning about the equal sign from a lesson that provided students with an opportunity to compare the equal sign with inequality symbols, a lesson that focused on the equal sign on its own, and a control lesson that did not focus on the equal sign.

For the purposes of this article, we focused on three critical aspects of mathematical knowledge: procedural knowledge, problem encoding, and conceptual knowledge (Hiebert, 1986). *Procedural knowledge* refers to the ability to execute action sequences for solving problems (including the ability to adapt these procedures for new problems) (Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler, & Alibali, 2001). *Problem encoding* refers to the internal mental representation of features of a problem (Chase & Simon, 1973; Kaplan & Simon, 1990; Siegler, 1976), which has been shown to be associated with problem-solving strategies (McNeil & Alibali, 2004). *Conceptual knowledge* refers to explicit or implicit understanding of principles that govern a domain and relations among pieces of information within or across a domain (Rittle-Johnson & Alibali, 1999; Tennyson & Cocchiarella, 1986).

We predicted that a lesson that involved comparing concepts (i.e., comparing the equal sign with other relational symbols) would lead to gains in conceptual understanding. Through comparison, students would acquire a deeper understanding of the meaning of the equal sign; for example, they would be able to provide an appropriate definition, recognize that it is similar to other relational symbols in mathematics, and judge nonstandard equation formats (e.g., $5 = 5$, $7 = 3 + 4$) as "making sense."

We expected that students, in gaining a better understanding of relational symbols via a comparison-based instructional approach, might also extend this knowledge to notice and encode the position of the equal sign when it appears in equations; for example, they would be able to accurately recreate equations from memory, including ones that do not have an ‘‘operations equals answer” structure (e.g., 3 + 4 + 6 = 3 + __).

Given that conceptual knowledge and procedural knowledge are intimately related, and that gains in one often lead to gains in the other (Rittle-Johnson & Alibali, 1999; Rittle-Johnson et al., 2001), we predicted that a better understanding of the equal sign through comparison with other relational symbols would also lead to gains in procedural knowledge as revealed by students' ability to more accurately solve problems that involve the equal sign (e.g., 3 + 4 + 6 = 3 + __).

In brief, we hypothesized that students who receive instruction about the equal sign would show greater gains in conceptual knowledge, equation encoding, and problem solving than students who do not receive such instruction. Furthermore, we hypothesized that students who compare the equal sign with other relational symbols would show greater gains in conceptual knowledge, equation encoding, and equation solving than students who learn about the equal sign alone. Finally, we hypothesized that students who compare the equal sign with other relational symbols would also better understand inequality symbols and improve their ability to solve inequality problems.

Past research has shown that comparison can foster procedural learning in mathematics (Rittle-Johnson & Star, 2007). The current study will help to determine whether or not comparison can also promote learning of mathematical *concepts*. This study further addresses whether comparing multiple concepts is more beneficial than learning a single concept in the same amount of time.

Method

Participants

A total of 112 third- and fourth-grade students (56 girls and 56 boys) were recruited from six urban parochial elementary schools serving predominantly Caucasian middle-class families. Of these students, 6 were excluded due to experimenter error, resulting in a final sample of 106 students. Students were randomly assigned to lesson groups. In the *comparing symbols* group, there were 23 girls and 20 boys (34 third graders and 9 fourth graders). In the *equal sign* group, there were 16 girls and 15 boys (23 third graders and 8 fourth graders). In the *control* group, there were 15 girls and 17 boys (23 third graders and 9 fourth graders). We did not ask students to report their racial/ethnic background. Traditional mathematics textbooks were used in mathematics classes for students in this sample (e.g., Greenes et al., 2005).

Procedure

Students participated for approximately half an hour in an individual session consisting of a pretest, a lesson, and a posttest. Prior to the session, students were randomly assigned to one of the three lesson conditions. A female experimenter conducted the sessions either on campus in a research laboratory or at students' schools in a small quiet room. In both settings, a video camera was placed to the side in an unobtrusive space within full view of the participants.

Overview

Students completed a pretest consisting of conceptual knowledge, problem encoding, and problem-solving tasks. Students then received a short lesson and were asked to work through 20 practice problems. Following the lesson, students completed a posttest that included all of the tasks in the pretest as well as additional tasks about inequalities. At no time during the pretest or posttest did the instructor provide feedback to students on their performance on any task. All of these tasks are described in further detail in the following sections.

Pretest

Students first completed a problem-solving test consisting of four equivalence problems. Two of the problems were in blank-final format (e.g., $a + b + c = a +$ __) and two were in blank-middle format (e.g., $a + b + c =$ __ $+ c$). For each problem, the experimenter asked students to solve the problem. After students produced a solution, the experimenter repeated that solution and asked students to explain with the prompt, "Can you tell me how you got that?" All students were able to produce solutions and explain them.

A reconstruction task was used to assess students' encoding of the structure of three blank-final equivalence problems (e.g., 3 + 5 + 6 = 3 + __). Past studies of students' encoding of equivalence problems (McNeil & Alibali, 2004) have shown that many students have difficulty in reconstructing such equations from memory, often converting them to an "operations equals answer" format (e.g., 3 + 5 + 6 + 3 = __), omitting the equal sign (e.g., 3 + 5 + 6 + 3), or making other sorts of errors in reconstructing the problem structure. Students viewed each equivalence problem for 5 seconds and then immediately reconstructed it by writing it on a blank notecard. Before beginning the task, students were reminded that they did not need to solve the problems but rather simply needed to write them down exactly as they saw them.

Next, the experimenter led students through three conceptual knowledge tasks adapted from Rittle-Johnson and Alibali (1999): a definition task, a symbol sort task, and a statement sort task.

In the *definition* task, the experimenter first ensured that students recognized the equal sign by asking "Can you tell me what this is?" while pointing at a card with only the equal sign on it. All students were able to name the equal sign. The experimenter then continued, "Can you tell me what it means?" After that, the experimenter asked, "And can it mean anything else?" to elicit a possible second definition.

The next task presented was the *symbol sort* task. Students were given a set of symbols printed on small cards; the set consisted of two operational symbols (+ and ×), three relational symbols (>, <, and =), and two numbers (4 and 9). Students were asked to place the symbols into three groups containing symbols that are similar according to the way we use them in mathematics.

The *statement sort* task involved students placing cards with 3 standard equations (e.g., 3 + 4 = 7) and 12 nonstandard equations (e.g., 7 = 7, 7 = 2 + 5) into piles that "make sense" and "don't make sense." Some of the standard and nonstandard equations were correct, and others were incorrect (e.g., 8 − 6 = 5, 5 + 3 = 5 − 3). Students were handed each equation card separately and were asked to sort it into one of the piles. As with the other pretest tasks, no feedback was provided. For the purpose of this study, we focus on students' sorting of the 8 nonstandard correct equations.

Lesson

Prior to the experimental session, students were randomly assigned to one of three lesson conditions. The three conditions differed in the type of lesson the students received: (a) comparing symbols, where students learned about the greater than sign, the less than sign, and the equal sign; (b) equal sign, where students learned only about the equal sign; and (c) control, where students learned a filler task.

In the comparing symbols lesson, the greater than, less than, and equal signs were placed in front of students as the experimenter explained, "All three of these symbols [pointing to each] are used in math to show a relationship between two quantities. They tell us how we can compare what is on either side of the symbol." The experimenter then went on to describe how each of the signs is used. The greater than sign, followed by the less than sign, and finally the equal sign were explained individually with the same amount of time being devoted to each symbol. After each symbol was introduced, the experimenter provided an example by presenting two cards: one with two addends and a plus sign in between and the other with a single number. Leaving a gap between the two cards, the experimenter explained, for example, "So if the first side is 4 + 3 and the second side is 2, when we compare the sides we see that 7 on this side is bigger or greater than 2 on this side, so we can put a greater than sign between them," as she pointed to each side and placed the appropriate symbol (in this case, a greater than sign) between the cards.

After describing all three symbols and providing an example for each, the experimenter engaged students in three more example problems. Each example problem contained a mathematical statement using one of the symbols, and students were asked to consider whether each statement made sense or not. For example, the experimenter said, "If you have 3 + 6 on this side and 7 on this side, would it make sense to put a greater than sign here to make this math sentence correct?" In the rare case that students had difficulty with the problem or produced an incorrect answer, the experimenter first asked students to solve the addition on one side and then compare the two sides by asking, "So is this number greater than, less than, or equal to the amount on the other side?" Feedback was provided until students produced a correct answer. Two of the three example problems were correct statements, and the third was incorrect.

Following the three example problems, students were asked to complete a sheet of practice problems on their own (see Table 1 for examples). On the practice sheet, students were asked to indicate whether 20 mathematical statements were correct or incorrect. The statements consisted of 10 inequalities and 10 equations, of which 10 statements were correct and 10 were incorrect. Both addition and subtraction statements were presented in random order. For each statement, students were asked to circle a "C" if the statement was correct or an "I" if it was incorrect. During the practice problems, no feedback was provided unless students needed clarification of the instructions.

The equal sign lesson followed a similar pattern except that students learned only about the equal sign in a lesson where time was equated with the comparing symbols lesson. Instead of learning about three different symbols as in the comparing symbols group, students in the equal sign group heard three different relational explanations of the equal sign. Each presentation of the equal sign used slightly different words and involved student participation as in the other conditions.

After each explanation, the experimenter led students through an example. Next, the experimenter led students through three more example problems (with feedback provided) with the format, "If there is a 3 + 6 on this side and a 9 on this side, would it make sense to put an equal sign here?" Again, as in the comparing symbols condition, two of the three example problems were correct and the third was incorrect. Students were then directed to identify 20 practice equations as correct or incorrect. The problems were the same as those used in the comparing symbols condition, with all inequality symbols changed to equal signs, yielding 10 correct statements and 10 incorrect statements (see Table 1 for examples). During the practice problems, no feedback was provided unless students needed clarification of the instructions.

For the control condition, a filler lesson and practice task were created. Students were directed to look at two cards (similar to those used in the other two conditions but with no symbol placed between them) and to determine which of the three numbers they saw was the biggest. The experimenter asked, "So if you have a 3 and a 4 on this side and a 6 on this side, which side has the biggest of all three numbers?" If students added the numbers, the experimenter reminded the students that they did not need to add them but rather only needed to choose the biggest of the three numbers. By the end of the initial three examples, no student had difficulty in producing correct answers. Thus, this filler task controlled for the length of time of the lesson as well as the focus on comparing one side with another side. The experimenter led students through three more examples

Table 1
Example practice problems.

Comparing symbols		Correct	Incorrect
$1 + 12 > 8$		C	I
$11 - 3 < 8$		C	I
$5 = 9 - 4$		C	I
Equal sign only		Correct	Incorrect
$1 + 12 = 8$		C	I
$11 - 3 = 8$		C	I
$5 = 9 - 4$		C	I
Control			
L	R	Left	Right
1 + 12	8	L	R
11 − 3	8	L	R
5	9 − 4	L	R

and then presented students with 20 practice problems (see Table 1 for examples). The same practice problems were used as in the other conditions except that one side of each mathematical statement was located in one column and the other side was located in a second column; there were no inequality symbols or equal signs present. Students were asked to find the largest of the three numbers and to locate which column it was in. Columns were marked "L" and "R" for left and right side, respectively. For each statement, students circled "L" if the biggest number was in the left column or "R" if it was in the right column.

Posttest

After completing the practice problems, all students went on to the posttest. The same tasks from the pretest were once again administered in the same order. The posttest items were structurally identical to those used on the pretest but used different numbers.

In addition, two tasks were used to assess inequality knowledge at posttest. These tasks were not presented at pretest because we did not want to provide students in the equal sign and control lesson groups with the opportunity to compare the equal sign with inequalities. The two tasks used at posttest were an inequality definition task, in which students were asked to define the greater than and less than signs, and an inequality problem-solving task, in which students were asked to complete four mathematical statements containing inequality symbols (e.g., 3 + 7 > 5 + __): two with greater than symbols and two with less than symbols. For each, students were asked to write a solution in the blank and provide an explanation for it. At the end of the experimental session, students who did not provide a correct solution to the last inequality problem were given a short lesson on how to solve inequality statements.

Students who participated at their schools were given a colorful pencil as a small reward for participating in the study. Students who participated in the laboratory were paid for their participation.

Coding

Codes for the conceptual knowledge, encoding, and problem-solving tasks were adapted from Rittle-Johnson and Alibali (1999).

Problem solving

Students' explanations of their solutions were transcribed and coded in terms of the strategies used to solve the problems. Correct strategies sometimes led to incorrect solutions because of arithmetic errors, so responses were coded in terms of whether or not the strategies were correct. Correct strategies (making both sides of the equation have the same amount) were given 1 point, and incorrect strategies (e.g., adding all of the numbers, adding the numbers before the equal sign) were given 0 points. There were four equivalence problems on each test, so each student received a score from 0 to 4 for pretest and posttest.

Encoding

Reconstructions were given 1 point if they contained the equal sign and plus signs in their correct positions (i.e., $a + b + c = a +$ __). Reconstructions with incorrect numbers or other errors that preserved the equivalence problem structure (e.g., missing a third addend on the left side of the equation: $a + b = a +$ __) were also given 1 point. All other reconstructions (e.g., converting to a standard equation: 3 + 5 + 6 + 3 = __; leaving out the equal sign: 3 + 5 + 6 + 3; leaving out the right plus sign: 3 + 5 + 6 = 3 __) were given 0 points. Because there were three problems on each encoding test, each student received a score from 0 to 3 for pretest and posttest.

Conceptual knowledge

Each of the three conceptual knowledge tasks was coded according to specific criteria for success as described below. Success scores for each of the three conceptual tasks were combined to form an overall conceptual understanding score ranging from 0 to 3 (a continuous measure similar to the problem-solving and encoding measures). This combined conceptual score, therefore, reveals the students' overall performance on the conceptual tasks (similar to the overall conceptual measures used in McNeil & Alibali, 2000, and Rittle-Johnson & Alibali, 1999).

For the definition task, students were considered as successful if their definitions showed evidence of a relational understanding of the equal sign (e.g., "It means the same as") and as unsuccessful if they did not (e.g., "It means the answer"). Students were considered as successful if they offered a relational definition after either the first or second prompt for a definition.

For the symbol sort task, students were considered as successful if they grouped the equal sign with other relational symbols (the greater than and less than signs) and as unsuccessful if they grouped the equal sign with the operation symbols.

For the statement sort task, students were considered as successful if they correctly identified all eight legitimate nonstandard equations as making sense (e.g., $7 = 7$, $7 = 3 + 4$, $3 + 2 = 6 - 1$) and as unsuccessful if they did not.

Inequality score

Inequality definitions were coded as successful if students' responses correctly identified which side of the statement would be greater than or less than the other side. To be considered as successful on the task, students were required to correctly define both the greater than sign and the less than sign. Students who confused the greater than and less than signs were also coded as correct. Students were considered as unsuccessful if they were not able to define both symbols.

Students' explanations for their solutions to the inequality problems were transcribed and coded as correct or incorrect. Students were considered as successful if they correctly solved all four inequality problems on the task and as unsuccessful if they did not.

Practice

Students' practice scores were calculated by summing the number of practice problems correctly recognized as correct or incorrect (or, in the control condition, the number of problems in which the correct side of the problem was marked as greater than the other side).

Reliability

Reliability was assessed on a subset of 20% of the students ($n = 22$) by a second coder who was blind to the conditions and purpose of the study. Percentage agreement between coders was 98% for coding problem-solving strategies for equivalence problems as correct or incorrect, 98% for coding reconstructions as correct or incorrect, 96% for coding equal sign definitions as correct or incorrect, 100% for coding inequality definitions as correct or incorrect, and 100% for coding problem-solving strategies for the inequality problems as correct or incorrect.

Results

The purpose of this study was to determine whether a lesson that provided an opportunity for comparison of the greater than, less than, and equal sign symbols would foster a greater relational understanding of the equal sign in elementary school students than a lesson that focused on the equal sign alone.

We used a multivariate analysis of variance (MANOVA) to test for significant differences among conditions at pretest—conceptual knowledge (definition, symbol sort, and statement sort), problem encoding, and problem solving—and found none (see Table 2 for pretest means and standard deviations by condition). We also determined that there were no differences between third- and fourth-grade students on any pretest measure. We then analyzed students' posttest performance in conceptual knowledge, encoding, and problem solving. Finally, we analyzed students' inequality understanding at posttest. We used linear regression to analyze continuous posttest measures and binary logistic regression to analyze categorical posttest measures.

We hypothesized that (a) the instructed groups would perform better than the control group and (b) the comparing symbols group would perform better than the equal sign group. To test these

Table 2
Unadjusted pretest and posttest means or proportions (and standard deviations) for each measure by condition.

	Pretest			Posttest		
	Comparing symbols	Equal sign only	Control	Comparing symbols	Equal sign only	Control
Problem solving (max = 4)	1.30 (1.39)	0.97 (1.56)	0.78 (1.73)	1.84 (1.84)	1.55 (1.84)	1.31 (1.65)
Encoding (max = 3)	1.28 (1.14)	1.45 (1.18)	1.22 (1.16)	1.98 (1.20)	1.90 (1.37)	1.63 (1.39)
Definition (max = 1)	0.42 (0.50)	0.35 (0.49)	0.31 (0.47)	0.77 (0.43)	0.68 (0.48)	0.44 (0.50)
Symbol sort (max = 1)	0.35 (0.48)	0.26 (0.44)	0.16 (0.37)	0.63 (0.49)	0.32 (0.48)	0.28 (0.46)
Statement sort (max = 1)	0.33 (0.47)	0.35 (0.49)	0.16 (0.37)	0.65 (0.48)	0.61 (0.50)	0.31 (0.47)

Note. Standard deviations are in parentheses. max, maximum.

Table 3
Effects (in standardized betas) of predictors on equal sign and inequality understanding at posttest.

	Instruction vs. control	Compare symbols vs. equal sign only	Pretest	Practice	Gender	Grade
Conceptual understanding	0.26**	0.19*	0.58**	0.15	0.04	0.04
Equal sign definition	0.65**	0.38	10.85	0.12	−0.21	1.08
Symbol sort	0.27	1.25**	12.57	0.34	1.15	0.71
Statement sort	0.65**	0.72*	3.63**	0.65**	0.49	0.21
Encoding	0.06	0.07	0.78**	0.00	−0.07	−0.02
Problem solving	0.02	0.00	0.75**	−0.05	0.09	0.11
Inequality definitions	0.61*	1.59**	NA	0.38	−0.57	−0.46
Inequality problem solving	0.30*	0.62*	NA	0.18	−0.68	0.61

Note. The process for obtaining standardized betas can yield values that are above or below the range of possible values; thus, the values should not be interpreted literally but rather should be used only in making group comparisons.
* $p < .05$.
** $p < .01$.

hypotheses, we used two contrasts to compare the three conditions for each outcome measure. The first contrast (instruction vs. control) compared the two lesson conditions with the control condition, and the second contrast compared the comparing symbols condition with the equal sign condition. Pretest scores (e.g., problem solving or encoding at pretest), practice scores, gender, and grade were included as predictors in the analyses as well (see Table 3 for a list of standardized coefficients for each predictor in the analyses). The reported means and proportions were adjusted to control for other predictors included in the model. Thus, the values presented in the text and in the figures represent hypothetical groups that are balanced or proportionately equal, having accounted statistically for any variance in pretest scores, practice scores, gender, and grade. The only exceptions are the unadjusted values presented in Table 2.

Equal sign conceptual understanding as a function of condition

Students displayed limited success on the conceptual tasks at pretest. In total, 38% of students provided a relational definition for the equal sign (8% after the first prompt and 30% after the second prompt). When asked to sort like symbols, however, 75% of students grouped the equal sign with operational symbols instead of with other relational symbols. Thus, even some of the students who provided a relational definition for the equal sign viewed the equal sign as ‘‘belonging” with operation symbols. On the statement sort task, although 100% of students were able to recognize 3 + 4 = 7 as a correct equation and 94% of students recognized 9 − 4 = 8 as an incorrect equation, only 28% of students successfully judged that the eight correct equations with nonstandard forms (e.g., 7 = 7, 7 = 3 + 4) make sense.

Because both the comparing symbols and equal sign lessons focused on the relational meaning of the equal sign, we anticipated that both lessons would lead to some gains in conceptual understanding; however, we expected greater gains in the comparing symbols group. We analyzed conceptual understanding of the equal sign at posttest between groups (see Fig. 1, first three columns). Students who received instruction had higher conceptual scores than students in the control group ($M = 2.99$, $SD = 0.98$ vs. $M = 2.09$, $SD = 1.00$), $\beta = 0.26$, $t(104) = 3.614$, $p < .001$, $d = .92$. Furthermore, students in the comparing symbols group had significantly higher conceptual scores than students in the equal sign group ($M = 3.22$, $SD = 0.92$ vs. $M = 2.68$, $SD = 1.02$), $\beta = 0.19$, $t(72) = 2.58$, $p = .01$, $d = .57$. Thus, the comparing symbols lesson promoted conceptual understanding more than the equal sign lesson. To better understand these posttest differences in conceptual understanding, we examined each conceptual task separately.

Equal sign definition

The first set of bars in Fig. 2 displays the proportion of students in each condition who produced relational definitions for the equal sign at posttest. As can be seen in the figure, students who received instruction about the equal sign were more likely to produce relational definitions than students who received no instruction (66% vs. 22%), $\beta = 0.65$, $z = 2.83$, $Wald(1, N = 106) = 8.01$, $p = .005$, $d = .61$. Students in the comparing symbols group were more likely than students in the equal sign group to produce relational definitions (74% vs. 57%), but this difference was not significant after controlling for other predictors, $\beta = 0.38$, $z = 1.06$, $Wald(1, N = 74) = 1.12$, $p = .29$, $d = .39$.

Symbol sort

The second set of bars in Fig. 2 displays the proportion of students in each condition who grouped the equal sign with other relational symbols at posttest. Students who received instruction were slightly, but not significantly, more likely than students in the control group to group the symbols appropriately (83% vs. 75%), $\beta = 0.27$, $z = 1.10$, $Wald(1, N = 106) = 1.21$, $p = .27$, $d = .17$. However, students in the comparing symbols condition were significantly more likely to group the equal sign

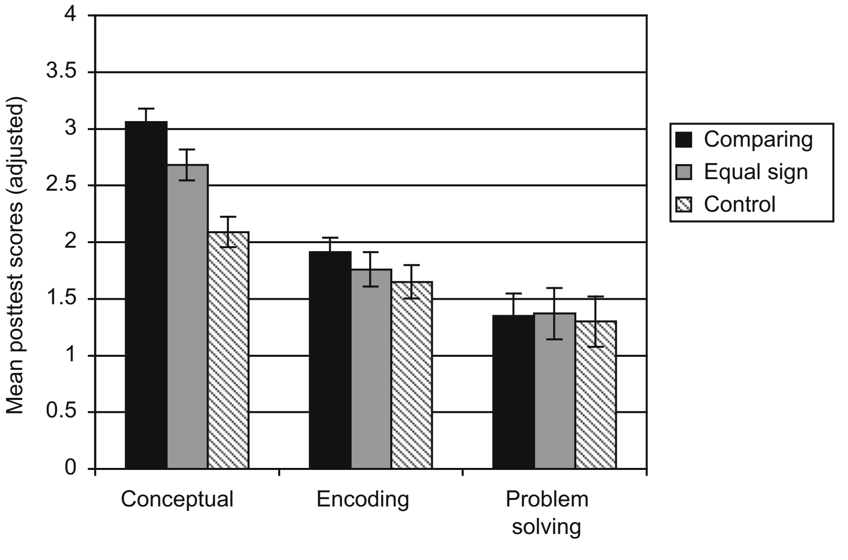

Fig. 1. Mean posttest scores for students in each condition for conceptual understanding, equation encoding, and problem solving. Means are adjusted for other predictors in the regression models. Error bars represent standard errors.

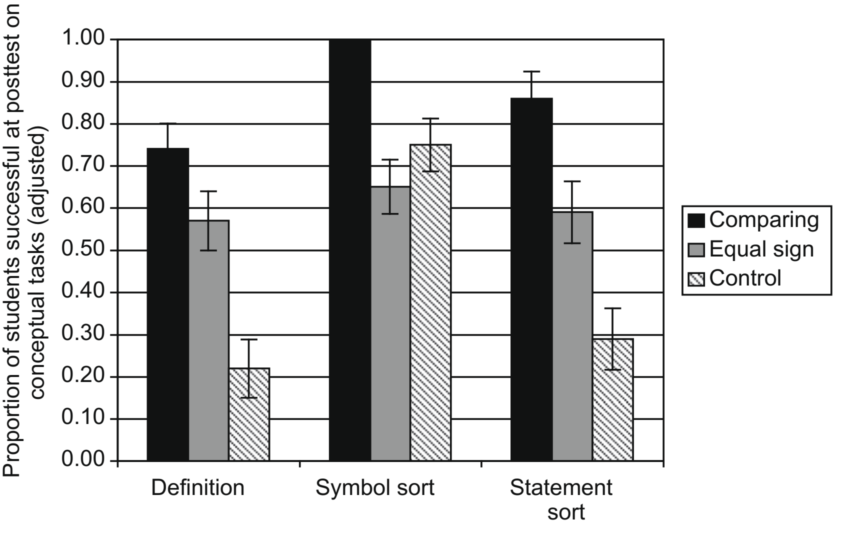

Fig. 2. Proportion of students in each condition who showed success at posttest in defining the equal sign, grouping the equal sign with relational symbols, and judging nonstandard equations as "making sense." Proportions are adjusted for other predictors in the models. Error bars represent standard errors.

with other relational symbols than students in the equal sign condition (100% vs. 65%), $\beta = 1.25$, $z = 2.79$, $Wald(1, N = 74) = 7.76$, $p = .005$, $d = .73$.

Statement sort

The third set of bars in Fig. 2 displays the proportion of students in each condition who succeeded on the statement sort task at posttest (i.e., who correctly judged all eight legitimate nonstandard equations as making sense). Students in the instruction conditions were more likely to succeed on this task than students in the control condition (72% vs. 29%), $\beta = 0.65$, $z = 3.04$, $Wald(1, N = 106) = 9.24$, $p = .002$, $d = .57$. Furthermore, students in the comparing symbols group were more likely to succeed than students in the equal sign group (86% vs. 59%), $\beta = .72$, $z = 1.97$, $Wald(1, N = 74) = 3.89$, $p = .049$, $d = .56$.

A final analysis of posttest performance on all three conceptual tasks revealed that the three tasks were positively correlated ($0.20 < rs < 0.33$ and $ps < .05$). Students who were successful on one conceptual task were likely to be successful on the other two conceptual tasks as well. For example, students who successfully sorted the equal sign with other relational symbols were also likely to successfully recognize legitimate nonstandard equations as making sense and to correctly define the equal sign as a relational symbol.

Overall, on the conceptual tasks, students in the instruction groups outperformed students in the control group. Students in the comparing symbols group outperformed students in the equal sign group both overall and on two of three conceptual tasks.

Equation encoding as a function of condition

Fully 80% of students produced inaccurate reconstructions of the equivalence problems in the encoding task at pretest. Of these students, 49% reconstructed at least one equivalence problem as a standard "operations equals answer" equation. The unfamiliarity of the equivalence problem structure (e.g., 3 + 5 + 6 = 3 + __) may have led students to misencode the position of the equal sign (e.g., 3 + 5 + 6 + 3 = __, 3 + 5 + 6 = __) (McNeil & Alibali, 2000, 2004).

We evaluated the number of problems correctly encoded at posttest. Fig. 1 presents the data for encoding in the second set of bars. Overall, students who received instruction correctly encoded more problems than students in the control group (M = 1.85, SD = 1.27 vs. M = 1.65, SD = 1.39), but this difference did not reach significance, β = 0.06, $t(104)$ = 0.98, p = .33, d = .15. Students' performance at posttest also did not differ significantly between the comparing symbols and equal sign groups (M = 1.91, SD = 1.20 vs. M = 1.76, SD = 1.37), β = 0.07, $t(72)$ = 1.09, p = .28, d = .12.

Overall, the groups did not differ in encoding at posttest. Although many students improved their encoding from pretest to posttest, this improvement did not vary by condition; it appears that exposure to equations, even in the absence of instruction, leads to improvements in equation encoding (Alibali, Phillips, & Fischer, 2009).

Problem solving as a function of condition

Although 100% of students were able to correctly solve standard equations (e.g., 3 + 4 + 6 + 3 = __), students' performance on the problem-solving task with nonstandard equations (e.g., 3 + 4 + 6 = 3 + __) at pretest was quite poor, with only 36% of students correctly answering *any* of the four equivalence problems. Solving equivalence problems was largely an "all or nothing" endeavor, with 80% of students using correct problem-solving strategies on either all or none of the four pretest problems. Students' strategies for solving the equivalence problems revealed that students applied their knowledge of standard addition problems when trying to solve new problems. Most students used incorrect strategies such as "add all" and "add to equal sign." Table 4 displays the proportion of students using each strategy at pretest and posttest.

We evaluated the number of problems solved using a correct strategy at posttest while controlling for pretest scores (refer to third set of bars in Fig. 1) as well as practice scores, gender, and grade. Overall, gains in problem solving from pretest to posttest were minimal, with fewer than one-third of students in each condition showing improvement. The number of equations solved by students using a correct strategy at posttest did not differ significantly among the three groups.

Understanding of inequalities as a function of condition

In addition to learning about the equal sign, the comparing symbols group learned about the greater than and less than signs in the same amount of time as the equal sign group learned about the equal sign alone. The foregoing analyses show that students in the comparing symbols group improved in their understanding of the equal sign at a level either on par with or better than students in the equal sign group, depending on the particular outcome measure considered. Did students in the comparing symbols group also gain in their understanding of inequalities? We assessed this aspect of students' knowledge only at posttest.

Defining inequality symbols

The first set of bars in Fig. 3 presents the proportion of students in each condition who accurately defined both the greater than and less than symbols. Overall, students who received instruction were more likely than students in the control group to accurately define both symbols (88% vs. 70%), β = 0.61, z = 2.49, $Wald(1, N = 106)$ = 6.20, p = .01, d = .50. As can be seen in the figure, this effect was driven by students in the comparing symbols condition. Indeed, students in the comparing symbols

Table 4
Proportion of students using each strategy on the problem-solving task at pretest and posttest.

	Description of error	Pretest	Posttest
Equalize	Make both sides of the equation have the same amount	0.23	0.41
Add all	Add all of the numbers	0.87	0.63
Add to equal sign	Add all of the numbers before the equal sign	0.65	0.53
Pattern	Continue a pattern in the addends	0.07	0.06
Add to equal and add all	Add all of the numbers of the equal sign, write that answer in the blank, add the solution to the last addend, write in another equal sign and blank sign, and write in final solution	0.02	0.01

Note. Proportions do not add to 1 because many students used two strategies.

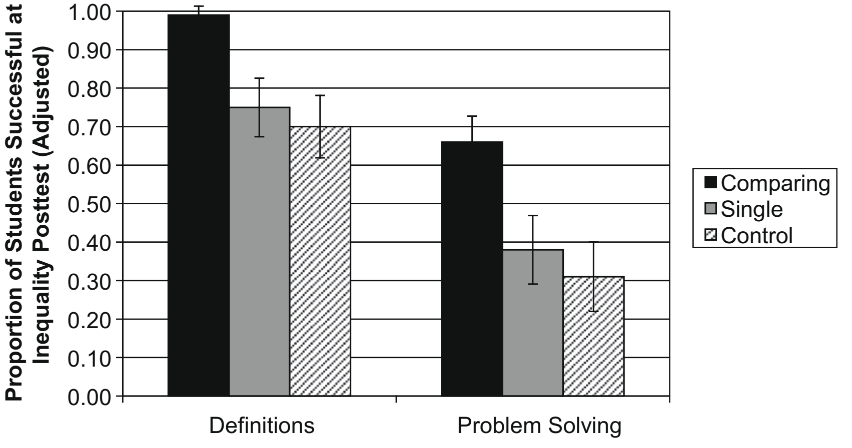

Fig. 3. Proportion of students in each condition who correctly defined inequality symbols and solved inequalities. Proportions are adjusted for other predictors in the models. Error bars represent standard errors.

condition were significantly more likely than students in the equal sign condition to accurately define both symbols (99% vs. 75%), $\beta = 1.59$, $z = 1.60$, $Wald(1, N = 74) = 6.75$, $p = .009$, $d = .79$.

Solving inequalities

We also examined students' performance at solving inequality problems. Students tended to use the same strategy when solving the inequality problems (91% of students used the same strategy for all four problems, i.e., all correct or all incorrect), and so the second set of bars in Fig. 3 presents the proportion of students who correctly solved all four inequality problems in each condition. Overall, students who received instruction were more likely to correctly solve the inequalities (55% vs. 31%), $\beta = 0.30$, $z = 1.97$, $Wald(1, N = 106) = 3.92$, $p = .048$, $d = .50$. Again, this effect was driven by students in the comparing symbols group, who were more likely than students in the equal sign group (68% vs. 38%) to correctly solve the inequalities, $\beta = 0.62$, $z = 2.23$, $Wald(1, N = 74) = 4.99$, $p = .025$, $d = .65$.

Overall, students in the comparing symbols condition who received a lesson in which they had the opportunity to compare the equal sign with inequality symbols showed greater gains in inequality understanding than students in the equal sign condition who received a lesson of the same length that was solely about the equal sign.

Discussion

Empirical summary

This study is the first to evaluate whether instruction that provides an opportunity for comparison is more effective for students' learning of a mathematical concept than instruction that does not foster comparison. As predicted, a comparing symbols lesson promoted better understanding of the equal sign in elementary school students than an equal sign lesson. More specifically, students who compared the equal sign with the greater than and less than symbols showed greater conceptual understanding of the equal sign at posttest than students who learned about the equal sign alone or not at all. In particular, students in the comparing symbols group were better able to group together relational symbols and were more likely to correctly recognize nonstandard equations as making sense. Students in the comparing symbols group and the equal sign group showed comparable performance at defining the equal sign, encoding equivalence problems, and solving equivalence problems.

Students in the comparing symbols group also showed greater understanding of inequality symbols than students in the equal sign group. They were better able to define the inequality symbols and to solve inequality problems. Overall, by comparing relational symbols, students gained deeper conceptual knowledge not only of the equal sign but also of inequality symbols.

How did comparison promote learning?

In this study, students in the comparing symbols group were more likely than other students to categorize the equal sign with other relational symbols at posttest. Thus, the lesson seemed to promote students' forming a category of symbols that express relations between quantities. It seems likely that students had learned, at some point in their elementary school education, that the equal sign conveys a relation; however, before the lesson, many students did not view this as a crucial feature of the equal sign. The comparing symbols lesson appeared to highlight "conveying a relation" as a key defining feature of the equal sign—one that could serve as a basis for deciding which symbols "go together" in the symbol sort task. Of course, it is possible that students' sorts may simply reflect the fact that the equal sign was presented with inequality symbols in the comparing symbols lesson; however, students' ability to sort relational symbols was also correlated with their ability to define the equal sign as a relational symbol and to recognize nonstandard equations as making sense. Perhaps the ability to group together relational symbols is a first step in understanding the similar meanings of these symbols.

When students performed inaccurately on the symbol sort task, they grouped the equal sign with operation symbols. This type of sorting may reflect students' experience with equations that contain the equal sign and at least one operation symbol. A comparison-based instructional technique may allow students to create new associations between the equal sign and other relational symbols and thereby promote students' forming a category of relational symbols and applying a relational interpretation to the equal sign. Students who have a weak understanding of relational symbols more generally may also benefit from comparing the equal sign with other relational symbols, strengthening their understanding of all types of relational symbols and not solely the equal sign.

Students in the comparing symbols group were also more likely than students in the equal sign group to correctly recognize nonstandard equations as making sense. Students who begin to view the equal sign as conveying a relation between quantities should be especially likely to recognize that equations are valid even if they do not follow the standard $a + b = c$ format. The lesson seemed to promote a more flexible understanding of equations.

Relatively few of the students in the comparing symbols group improved in solving or encoding equivalence problems, and the same was true for students in both of the other groups. We know that students who hold a relational understanding of the equal sign are more likely to correctly solve equivalence problems (Knuth et al., 2006); therefore, we anticipated gains in both procedural and conceptual knowledge. The two forms of knowledge are closely linked; indeed, the use of correct procedures is sometimes taken as an indicator of conceptual understanding (Prather & Alibali, 2009). Despite these close ties, however, it may be that when comparing procedures, gains in procedural knowledge are more likely (e.g., Rittle-Johnson & Star, 2007), and when comparing exemplars of a concept, gains in conceptual knowledge are more likely, as seen in this study.

It is possible that, for a conceptual lesson to lead to gains in encoding and procedural knowledge, a stronger or lengthier manipulation may be required. Alternatively, more time for processing or knowledge integration may be needed. With time, the consolidation of conceptual knowledge could lead to better encoding and to generation of correct problem-solving strategies.

It is also possible that students demonstrated poor encoding and problem-solving performance because the symbolic equivalence problems elicited procedural schemas for solving the problems that were resistant to change in response to newly gained conceptual knowledge. For many students, the "add all" strategy is quite strongly entrenched and difficult to let go of (McNeil & Alibali, 2005b). Measures of encoding and problem solving in nonsymbolic formats (Sherman & Bisanz, 2009) might reveal greater knowledge because they may be less likely to recruit entrenched incorrect approaches.

In the comparing symbols lesson, students were provided with an opportunity to compare the equal sign with other relational symbols. Students were invited to make this comparison through symbols presented together and through mixed practice with inequalities and equations. Although students were given ample opportunity to compare and contrast the ways in which these symbols were used, it is not clear exactly *how* students actually compared the symbols. Comparison could have occurred when students were informed of the meaning of each symbol and realized that the equal sign was similar to the greater than and less than symbols. Students with a more abstract general concept of relational symbols may have compared the equal sign not with the individual inequality symbols specifically but rather with their general concept of relational symbols, perhaps adding the equal sign to this abstract category. Comparisons could have been made later as well when students compared the use of inequality symbols with the use of the equal sign in solving problems using the symbols. Because the lessons simply allowed the comparisons to be made and did not explicitly compare the symbols for the students, it is not clear which type of comparison was most beneficial: comparing one exemplar of a concept with another, comparing an example with the overarching concept, or comparing one problem with another problem. This study invites further examination of how different types of comparisons may differentially foster learning.

Educational implications

Our findings suggest that a comparison-based approach to instruction can be applied in instruction about mathematical concepts; indeed, it may be particularly beneficial for concept learning. Past studies of instructional approaches that serve to highlight comparison, such as contrasting cases instruction, have not focused on learning of mathematical concepts. A few past studies have assessed the effects of comparison on conceptual understanding when learning a mathematical procedure; however, the results are mixed because few studies have used measures that are appropriate to gauge conceptual understanding. The current study provides empirical support for the notion that comparison can indeed facilitate learning of mathematical concepts.

Lessons that direct students' attention to similarities between previously learned exemplars of a concept and a target concept may help students to discern key features of the examples and, in so doing, to gain a deeper understanding of both the known concept and the target concept. In this study, through a lesson comparing inequality symbols with the equal sign, students improved their understanding of both types of symbols and appeared to form one larger category of relational symbols that ties together the common aspects of inequality symbols and the equal sign. Although a lesson focusing on the equal sign alone may seem direct and easy for students to grasp, a lesson comparing the equal sign with inequalities allowed students to better understand the three symbols as related concepts and, at the same time, to better understand each symbol individually.

Limitations

Some limitations of this study must be acknowledged. First, the study was conducted in a one-on-one instructional setting. Future research will be needed to determine whether the findings "scale up" to classroom lessons or other sorts of instructional settings.

Second, we did not find beneficial effects of comparing symbols instruction on equation encoding or equivalence problem solving as we had expected. It is possible that if we had tested students' problem solving at a later time, such effects might have been observed. Future research should look in more detail at change in knowledge *over time* as a function of instruction that involves comparison. Future research should also use outcome measures in nonsymbolic formats because such measures may be more likely to reveal subtle changes in learners' knowledge than measures in symbolic format such as the ones we used in this study.

Third, the comparing symbols lesson was not beneficial for every student. Despite the one-on-one lesson, some students still did not display a relational understanding of the equal sign at posttest. It may be that, like other types of instruction, comparing symbols is most beneficial for students whose incorrect operational view of the equal sign is less deeply entrenched (McNeil & Alibali, 2005b). Future research is needed to test this hypothesis.

Conclusion

Despite these limitations, the current findings are important because they show that comparison can promote understanding of mathematical *concepts.* Specifically, a lesson in which elementary school students compared inequality symbols and the equal sign facilitated a relational understanding of the equal sign more than a lesson in which students learned about the equal sign alone in the same amount of time. Thus, a comparison-based instructional method can be of value in teaching and learning mathematical concepts.

References

Alibali, M. W., Phillips, K. M. O., & Fischer, A. D. (2009). Learning new problem-solving strategies leads to changes in problem representation. *Cognitive Development, 24*, 89–101.

Baroody, A., & Ginsburg, H. (1983). The effects of instruction on children's understanding of the "equals" sign. *Elementary School Journal, 84*, 199–212.

Behr, M. J., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching, 92*, 13–15.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in the elementary school*. Portsmouth, NH: Heinemann.

Chase, W. G., & Simon, H. A. (1973). Perception in chess. *Cognitive Psychology, 4*, 55–81.

Gentner, D., & Markman, A. B. (1994). Structural alignment in comparison: No difference without similarity. *Psychological Science, 5*, 152–158.

Gentner, D., & Medina, J. (1998). Similarity and the development of rules. *Cognition, 65*, 263–297.

Greenes, C., Larson, M., Leiva, M. A., Shaw, J. M., Stiff, L., Vogeli, B. R., et al (2005). *Houghton Mifflin math*. Boston: Houghton Mifflin.

Grobman, K. H., & Alibali, M. W. (2007). *Understanding the equal sign and learning algebra.* Paper presented at the biennial meeting of the Society for Research in Child Development, Boston.

Hiebert, J. (1986). *Conceptual and procedural knowledge: The case of mathematics*. Hillsdale, NJ: Lawrence Elrbaum Associates.

Kaplan, C. A., & Simon, H. A. (1990). In search of insight. *Cognitive Psychology, 22*, 374–419.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317–326.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 390–419). New York: Macmillan.

Kliman, M., Tierney, C., Russell, S. J., Murray, M., & Akers, J. (1997). *Investigations in number, data, and space*. Palo Alto, CA: Dale Seymour.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*, 297–312.

Kotovsky, L., & Gentner, D. (1996). Comparison and categorization in the development of relational similarity. *Child Development, 67*, 2797–2822.

Li, X., Ding, M., Capraro, M. M., & Capraro, R. M. (2008). Sources of differences in children's understandings of mathematical equality: Comparative analysis of teacher guides and student texts in China and the United States. *Cognition and Instruction, 26*, 195–217.

McNeil, N. M. (2007). U-shaped development in math: 7-year-olds outperform 9-year-olds on equivalence problems. *Developmental Psychology, 43*, 687–695.

McNeil, N. M., & Alibali, M. W. (2000). Learning mathematics from procedural instruction: Externally imposed goals influence what is learned. *Journal of Educational Psychology, 92*, 734–744.

McNeil, N. M., & Alibali, M. W. (2004). You'll see what you mean: Students encode equations based on their knowledge of arithmetic. *Cognitive Science, 28*, 451–466.

McNeil, N. M., & Alibali, M. W. (2005a). Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of Cognition and Development, 6*, 285–306.

McNeil, N. M., & Alibali, M. W. (2005b). Why won't you change your mind? Knowledge of operational patterns hinders learning and performance on equations. *Child Development, 76*, 883–899.

McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. C., Hattikudur, S., et al (2006). Middle-school students' understanding of the equal sign: The books they read can't help. *Cognition and Instruction, 24*, 367–385.

Perry, M., Church, R. B., & Goldin-Meadow, S. (1988). Transitional knowledge in the acquisition of concepts. *Cognitive Development, 3*, 359–400.

Prather, R. W., & Alibali, M. W. (2009). Development of arithmetic principle knowledge: How do we know what learners know? *Developmental Review, 29*, 221–248.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*, 175–189.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*, 346–362.

Rittle-Johnson, B., & Star, J. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*, 561–574.

Schwartz, D. L., & Bransford, J. D. (1998). A time for telling. *Cognition and Instruction, 16*, 475–522.

Seo, K.-H., & Ginsburg, H. P. (2003). "You've got to carefully read the math sentence...": Classroom context and children's interpretations of the equals sign. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills*. Mahwah, NJ: Lawrence Erlbaum.

Sherman, J. L., & Bisanz, J. (2009). Equivalence in symbolic and nonsymbolic contexts: Benefits of solving problems with manipulatives. *Journal of Educational Psychology, 101*, 88–100.

Siegler, R. S. (1976). Three aspects of cognitive development. *Cognitive Psychology, 8*, 481–520.

Star, J. R., & Rittle-Johnson, B. (2009). It pays to compare: An experimental study on computational estimation. *Journal of Experimental Child Psychology, 102*, 408–426.

Tennyson, R. D., & Cocchiarella, M. J. (1986). An empirically based instructional design theory for teaching concepts. *Review of Educational Research, 56*, 40–71.