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Learning new problem-solving strategies leads to changes in problem representation☆

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ABSTRACT

Children sometimes solve problems incorrectly because they fail to represent key features of the problems. One potential source of improvements in children's problem representations is learning new problem-solving strategies. Ninety-one 3rd- and 4th-grade students solved mathematical equivalence problems (e.g., $3+4+6=3+\_\_$) and completed a representation assessment in which they briefly viewed similar problems and either reconstructed each problem or identified it in a set of alternatives. Experimental groups then received a lesson about one or both of two solution strategies, the equalize strategy and the add–subtract strategy. A control group received no instruction. All children completed posttest assessments of representation and problem solving. Children taught the equalize strategy improved their problem representations more than those not taught it. This pattern did not hold for the add–subtract strategy. These results indicate that learning new strategies is one source of changes in problem representation. However, some strategies are more effective than others at promoting accurate problem representation.

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With development, children often shift from using incorrect or inefficient strategies for solving problems to using correct or more efficient strategies. This shift has been observed in many domains, including conservation, categorization, memory, and arithmetic, to name a few. Understanding the mechanisms by which children abandon old strategies in favor of newer, better strategies is one of the fundamental challenges in the modern study of cognitive development (Siegler, 1996).

One reason children may use incorrect or inefficient strategies to solve problems is that they fail to accurately represent key features of the problems. In the present context, *problem representation* refers to "the internal depiction or re-creation of a problem in working memory during problem solving" (Rittle-Johnson, Siegler, & Alibali, 2001, p. 348). Problem representations incorporate both features of a presented problem that the solver encodes, and knowledge from long-term memory (e.g., examples of similar problems) that is activated. Children form inaccurate or incomplete problem representations for many different types of problems, including balance scale problems (Siegler, 1976; Siegler & Chen, 1998), arithmetic word problems (Dean & Malik, 1986), probability problems (Dean & Mollaison, 1986), proportional reasoning problems (Fujimura, 2001), serial recall problems (McGilly & Siegler, 1990) and mathematical equivalence problems (McNeil & Alibali, 2004).

Past research suggests close ties between problem representation and strategy use (McNeil & Alibali, 2004; Siegler, 1976). Relations between problem representation and strategy use are likely bidirectional, in the sense that problem representation affects strategy use and strategy use guides representation. When solvers represent particular problem features, they are in a position to generate or apply strategies that depend on those features. Indeed, when children begin to represent new problem features, they sometimes generate new strategies and abandon existing ones (Alibali, McNeil, & Perrott, 1998; Rittle-Johnson et al., 2001; Siegler, 1976). For example, Siegler identified children who used a weight-only strategy to make judgments on a balance scale task (predicting which side of the scale would go down if props under the scale were removed). He then taught children to encode the distance of the weights from the fulcrum, as well as the number of weights. After this training, many children began to use strategies that involved both weight and distance. Thus, children improved their problem representations and consequently generated a more sophisticated problem-solving strategy.

Knowledge of problem-solving strategies also presumably affects the features of problems that solvers represent. Indeed, past studies have documented differences in problem representation between individuals with more knowledge and those with less knowledge in various domains, including physics (Chi, Feltovich, & Glaser, 1981; Larkin, 1983; Larkin, McDermott, Simon, & Simon, 1980), chess (Chase & Simon, 1973), and mathematical equivalence (McNeil & Alibali, 2004; Rittle-Johnson & Alibali, 1999), suggesting that knowledge affects problem representation. However, to date, all of the evidence on this point is correlational rather than experimental; no studies have shown that gains in knowledge actually *cause* changes in problem representation. The aim of the present study is to provide causal evidence on this point. The study also provides evidence regarding the conditions under which gains in knowledge lead to improvements in problem representation. We focus in particular on gains in strategy knowledge and on conditions in which such knowledge is directly instructed.

Children are frequently taught new problem-solving strategies in educational settings, particularly in academic domains such as mathematics. The purpose of the present study is to test whether learning new strategies from direct instruction leads to changes in children's problem representations. We hypothesize that when children learn new strategies, they may start to attend to problem features that they had not previously encoded, but that are important for implementing those new strategies. One implication of this idea is that when children who use incorrect strategies learn new, correct strategies, their problem representations should improve.

We investigate this issue using mathematical equivalence problems, which are equations with addends on both sides of the equal sign, e.g., $3 + 4 + 6 = 3 + \_\_$. Research has shown that many elementary school students solve such problems incorrectly and that there are several different correct strategies that students can readily learn from brief instruction (Alibali, 1999; Cook & Goldin-Meadow, 2006; McNeil & Alibali, 2004). Moreover, many elementary school students represent equivalence problems inaccurately; for example, many students fail to notice the addend on the right-hand side of the equation (McNeil & Alibali, 2004). This leads to the possibility that improvements in encoding of the problems could be observed within an experimental session. Thus, equivalence problems are an ideal type of problem for investigating the relation between problem representation and strategy use.

Because there is more than one correct strategy for solving equivalence problems, such problems also afford the opportunity to test whether different correct strategies had similar, positive effects on problem representation. We chose to provide instruction about two different commonly-used strategies: *equalize*, which involves finding a solution that makes the two sides of the equation sum to the same quantity (typically by trial and error or by recalling arithmetic facts), and *add–subtract*, which involves summing the numbers on the left side and subtracting the number on the right. In natural settings, mathematics lessons sometimes focus on a single approach to solving problems and sometimes focus on multiple approaches. In this experiment, we instructed children on these two strategies using a 2×2 design, so that we could also examine effects of single versus multiple strategy instruction.

Past studies that sought to characterize solvers' problem representations have utilized problem reconstruction as a measure of problem representation (Chase & Simon, 1973; Siegler, 1976). For example, Siegler asked children to view a balance scale for a brief period of time and then to reconstruct the configuration of the balance scale on a different, empty balance scale. On the basis of children's reconstructions, he inferred whether children represented both the number of disks on each peg, and the distance of each set of disks from the fulcrum. Building on this work, in the present study, we also utilize a reconstruction task, in which we ask children to view a mathematical equivalence problem for 5 s and then to reconstruct the problem by writing it on a note card. This task has been used extensively in work on mathematical equivalence (McNeil & Alibali, 2000, 2004; Rittle-Johnson & Alibali, 1999). In this study, we also utilize a recognition version of this task, in which we asked children to view a mathematical equivalence problem for 5 s and then to select that problem from a set of alternatives. This recognition task has also been used as a measure of problem representation (McNeil & Alibali, 2005).

1. Method

1.1. Participants

Participants were 91 elementary school students, 22 girls and 16 boys in 3rd grade and 33 girls and 20 boys in 4th grade. Children ranged in age from 8–6 to 10–7 years, with a mean age of 9–8 years (SD = 7.1 months). Participants were recruited from public and parochial schools in a mid-sized city in the midwestern USA. One child chose not to complete the session because he wished to participate in an upcoming classroom activity, and one child did not complete the transfer test due to time constraints.

1.2. Procedure

Each child participated individually in an experimental session lasting approximately half an hour. The session consisted of six parts: a problem-solving pretest, a problem representation pretest, a lesson about a strategy (or strategies), a problem representation posttest, a problem-solving posttest, and a transfer test. Except for some of the transfer problems, all equivalence problems presented the blank in final position (e.g., $3+4+5=3+\_\_$).

Problem-solving pretest. In the problem-solving pretest, children worked with the experimenter at a white board. Children were asked to solve three equivalence problems, which have addends on both sides of the equal sign (e.g., $3+4+6=3+\_\_$), and two "standard" addition problems (e.g., $3+5+7+3=\_\_$), which have the equal sign and answer blank at the end of the problem. The standard problems were included because they are more typical of children's everyday mathematical experiences, so children could feel confident about solving them. Such problems might also activate children's "typical addition" schema, and in doing so, increase the likelihood that children would represent the equivalence problems as standard problems (McNeil & Alibali, 2005). The standard problems were presented as the second and fourth problems, interspersed among the equivalence problems. Immediately after solving each of the problems, children were asked to explain their solutions.

Problem representation pretest. The problem representation pretest consisted of two segments, administered in fixed order. In the *reconstruction* segment, children were presented with three equivalence problems, printed in 72-point font on 8.5 in. × 11 in. paper, for 5 s each. After viewing each problem, children were asked to write the problem on a note card "exactly as [you] saw it". In the

$$5 + 7 + 8 + 5 = \_\_$$

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$$5 + 7 + 8 = \_\_$$

$$5 + 7 + 8 = 5\ \_\_$$

Fig. 1. Sample recognition trial from the representation task. On this trial, the experimenter presented the problem $5+7+8=5+\_\_$ for 5 s, then covered the problem, and gave the participant a sheet of paper containing the alternatives shown. The task was to circle the presented problem.

recognition segment, children were presented three equivalence problems, also printed in 72-point font on 8.5 in. × 11 in. paper, for 5 s each. After each problem was presented and removed, children were asked to select it from a set of seven alternatives. The incorrect alternatives were based on errors made by children in previous studies; all were errors in representing the structure of the problem (*conceptual* errors, described in more detail below). A sample trial from the recognition segment is presented in Fig. 1.

Lesson. For the lesson, children were randomly assigned to one of four experimental conditions, based on a 2 × 2 design. The two lessons focused on common correct strategies, the equalize strategy and the add–subtract strategy. Thus, the 2 (equalize instruction: yes or no) × 2 (add–subtract instruction: yes or no) design yielded four experimental conditions: no instruction (control), equalize instruction only, add–subtract instruction only, and both equalize and add–subtract instruction.

In each lesson, children were presented three additional equivalence problems. Children in the equalize, add–subtract, and both-equalize-and-add–subtract conditions received instruction about the target strategy (or strategies) with each of the three problems. For each problem, instruction was provided before the child attempted to solve the problem. Children in the no-instruction condition received no instruction, but simply solved the three problems. No feedback was given to children in any condition regarding whether their solutions were correct. We chose not to offer feedback about correctness so as to be sure that any changes in representation we observed were due to the strategy instruction rather than feedback.

For the equalize strategy lesson, the experimenter said:

> Now I'm going to tell you a way to solve this problem. First, add up this side (tracing gesture under left side). Then, find a number to go in the blank that will make this side (tracing gesture under right side) add up to that same number. And that's your answer. Let me just repeat that. First, add up this side (tracing gesture under left side). Then, find a number to go in the blank that will make this side (tracing gesture under right side) add up to that same number. And that's your answer. OK, can you go ahead and solve it? I'm not going to tell you whether you're right or wrong; I'd just like you to solve it and tell me what you did.

For the add–subtract strategy lesson, the experimenter said:

> Now I'm going to tell you a way to solve this problem. First add up this side (tracing gesture under left side). Then, subtract this number (point to number on right side). And that's your answer. Let me

Table 1
Commonly used strategies for solving the problems and sample verbal responses.

Strategy	Example
Incorrect strategies	
Add all	I added 3 plus 4 plus 6 plus 3 and I got 16
Add to equal sign	3 plus 4 plus 6 is 13
Correct strategies	
Equalize	3 plus 4 plus 6 is 13, and I needed to figure out what plus 3 is 13, and it's 10
Add–subtract	I added 3 and 4 and 6 and then I subtracted the other 3
Group	There was a 3 here and 3 here, so I just added 4 plus 6

Note: All examples refer to the problem 3 + 4 + 6 = 3 + __.

> just repeat that. First add up this side (tracing gesture under left side). Then, subtract this number (point to number on right side). And that's your answer. OK, can you go ahead and solve it? I'm not going to tell you whether you're right or wrong; I'd just like you to solve it and tell me what you did.

In both the equalize and-add–subtract condition, the experimenter described the equalize strategy and then said, "Another way you can solve it is to do this," and then described the add–subtract strategy. We did not offer any information about the conceptual connections between the two strategies, because our focus was on gains in strategy knowledge, rather than gains in conceptual knowledge.

Problem representation posttest. Following the lesson, children completed a problem representation posttest that was comparable to the problem representation pretest and that included both reconstruction and recognition segments.

Problem-solving posttest. Next, children solved and explained three more equivalence problems and two more standard addition problems in a posttest.

Transfer test. Finally, children were given a paper-and-pencil transfer test that included problems that varied in the position of the blank, the presence of a repeated addend, and the operation (addition and multiplication). The problems used on the transfer test are presented in Appendix A. Note that the first problem on the transfer test was structurally identical to those used in the main body of the experiment. The other transfer problems were all relatively near transfer problems, because they could be solved using relatively minor adaptations of the taught strategies.

1.3. Coding

Problem solving. We coded the problem-solving strategies that children expressed in their verbal explanations. Coding definitions and examples of explanations conveying common correct and incorrect problem-solving strategies are presented in Table 1. Children received one point for each problem solved with a correct strategy. Thus, children could receive a score of 0 to 3 for each of the three problem-solving segments (pretest, lesson, and posttest).

Problem representation. For the reconstruction segment of the representation assessments (three items), we first evaluated whether children's problem reconstructions were correct or incorrect. Errors were further classified as *number* errors, which involved incorrectly reconstructing the specific numbers or the order of the numbers in the problems (e.g., for the equation 4 + 9 + 3 = 4 + __, writing 4 + 9 + 3 = 5 + __), or *conceptual* errors, which involved either converting the equivalence problems to standard addition problems (e.g., for the equation 4 + 9 + 3 = 4 + __, writing 4 + 9 + 3 + 4 = __), or producing syntactically incorrect equations (e.g., for the equation 4 + 9 + 3 = 4 + __, writing 4 + 9 + 3 + 4 or 4 + 9 + 3 = 4 __). Some sample problem reconstructions that contain conceptual errors are presented in Table 2. We calculated a reconstruction score for each child, which was the number of trials free of conceptual errors (pretest $M = 1.61$, SE = 0.13; posttest $M = 2.43$, SE = 0.01).

For the recognition segment (three items), we scored the number of correct choices (pretest $M = 1.67$, SE = 0.11; posttest $M = 2.38$, $SE = 0.10$). Reconstruction and recognition scores were significantly correlated at both pretest and posttest (pretest $r(90) = 0.34$, $z = 3.31$, $p < 0.001$; posttest $r(90) = 0.46$, $z = 4.67$, $p < 0.0001$).

Table 2
Common conceptual errors on the reconstruction task.

Error type	Example
Convert to standard equation	3 + 4 + 6 + 3 = __
No equal sign	3 + 4 + 6 + 3
No plus sign or blank on right side of equation	3 + 4 + 6 = 3
No plus sign on right side of equation	3 + 4 + 6 = 3 __
Extra equal sign	3 + 4 + 6 = 3 = __

Note: All examples refer to the problem 3 + 4 + 6 = 3 + __.

Finally, we calculated a total representation score for each child, which was the number of trials free of conceptual errors on the reconstruction task plus the number of correct choices on the recognition task. Recall that all of the incorrect choices on the recognition task were conceptual errors; therefore, this combined score provides a measure of the number of conceptual errors produced by each child across both the reconstruction and the recognition tasks.[1] Cronbach's alpha for this six-item assessment of problem representation was 0.73 at pretest and 0.79 at posttest.

Transfer test. We evaluated children's problem solutions for the transfer test. Each child received one point for each solution that was exactly correct.[2] Because the multiplication problems require a more generalized understanding of equivalence than the addition transfer problems, we scored and analyzed the addition and multiplication transfer problems separately. The first addition problem on the transfer test was identical in structure to the problems on the pretest and posttest; this problem was not included when calculating addition transfer scores, so those scores ranged from 0 to 3. Multiplication transfer scores ranged from 0 to 4.

Reliability of coding procedures. To establish reliability of the coding procedures, a second coder rescored a randomly selected subset of 15 participants for problem solving and 15 participants for reconstructions. Agreement was 95% for categorizing problem-solving strategies expressed in speech and 97% for coding reconstructions as either correct, incorrect with a number error, or incorrect with a conceptual error.

2. Results

2.1. Relation between problem representation and strategy use at pretest

If the representation assessment provides a good measure of participants' problem representations, participants who used correct strategies on the problem-solving pretest should have higher scores on the representation pretest than participants who used incorrect strategies. This was indeed the case, as seen in Fig. 2. Pretest problem-solving scores significantly predicted pretest representation scores, $\beta = 0.45$, $t(88) = 4.75$, $p < 0.001$, and they explained a significant proportion of the variance in pretest representation scores, $r^2 = 0.20$, $F(1,88) = 22.56$, $p < 0.001$. Thus, use of correct strategies was associated with more accurate problem representation.

The joint distribution of pretest representation scores and pretest problem-solving scores is presented in Table 3. Pretest problem-solving scores had a U-shaped distribution, with most students solving either none (64%) or all (22%) of the problems correctly. In contrast, pretest representation

[1] One of the incorrect alternatives on the recognition test was in fact a correct equation with addends on both sides, but with one of the addends missing (e.g., 5 + 7 = 8 + __ in Fig. 1). This alternative could be considered conceptually correct; however, it was chosen exceedingly rarely (only once by one child on the pretest and once by one other child on the posttest), so the results do not change depending on whether this choice was treated as correct or incorrect.

[2] For the transfer test, we counted as correct only solutions that were exactly correct. We did so because arithmetic errors were difficult to identify, since we did not collect verbal explanations for the transfer problems. For example, for the transfer problem $2 \times 4 \times 3 = \_\_ \times 6$, the correct answer is 4. For a child who answered 5, it is impossible to tell whether the child's solution represents an arithmetic error with a correct solution strategy (e.g., dividing 24 by 6 and getting 5 rather than 4) or an incorrect solution strategy (e.g., adding 2 and 3).

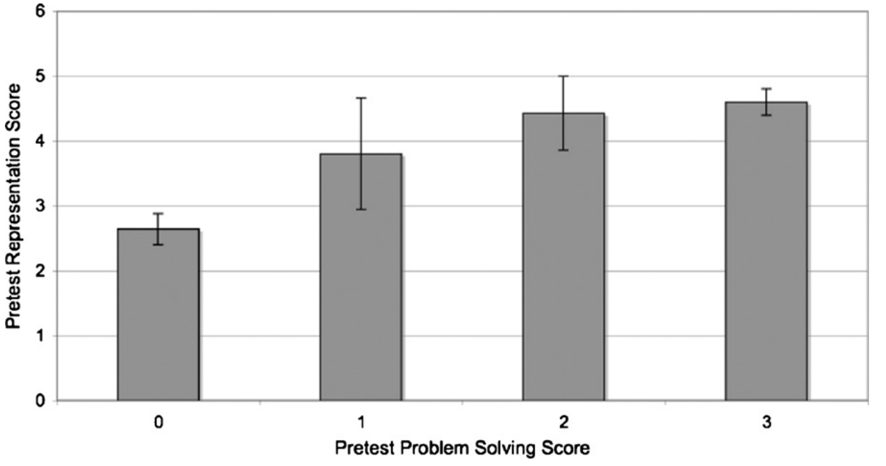

Fig. 2. Mean score on the pretest representation assessment (of six) as a function of pretest problem solving score.

scores were more evenly distributed, with between 9 and 20% of participants receiving each possible score from 0 to 6. A handful of students ($N = 4$) performed perfectly on the pretest representation assessment, despite solving none of the pretest problems correctly. However, the reverse did not hold; among students who solved all of the pretest problems correctly, all scored 2 or better on the representation assessment.

Because our main goal was to examine whether learning new, correct strategies influenced problem representation, we limited our analyses of strategy and representation change to participants who used incorrect strategies to solve all three pretest equivalence problems and who did not perform at ceiling (i.e., six of six correct) on the pretest representation assessment. These participants ($N = 54$) could both learn new strategies and improve their representations. For analyses of transfer performance, we included the full sample; however, one child did not complete the transfer test, so the N for the analyses of transfer performance is 89 rather than 90.

2.2. Problem-solving performance

Our main goal was to examine whether learning new problem-solving strategies led to improvements in problem representation. However, we first needed to establish that participants in the experimental groups did indeed learn new strategies. We examined performance on equivalence problems both for the lesson and the posttest. The data were analyzed using 2 (time point: lesson or posttest) × 2 (equalize instruction: yes or no) × 2 (add–subtract instruction: yes or no) analysis of variance (ANOVA), with number of problems on which a correct strategy was used as the dependent

Table 3
Number of children who received each combination of pretest problem-solving score and pretest representation score.

	Pretest problem-solving score				
Pretest representation score	0	1	2	3	Total (proportion of total)
0	9	0	0	0	9 (0.10)
1	7	1	0	0	8 (0.09)
2	14	0	1	3	18 (0.20)
3	9	1	1	2	13 (0.14)
4	8	1	1	3	13 (0.14)
5	7	1	2	4	14 (0.16)
6	4	1	2	8	15 (0.17)
Total (proportion of total)	58 (.64)	5 (.05)	7 (.08)	20 (.22)	90 (1.00)

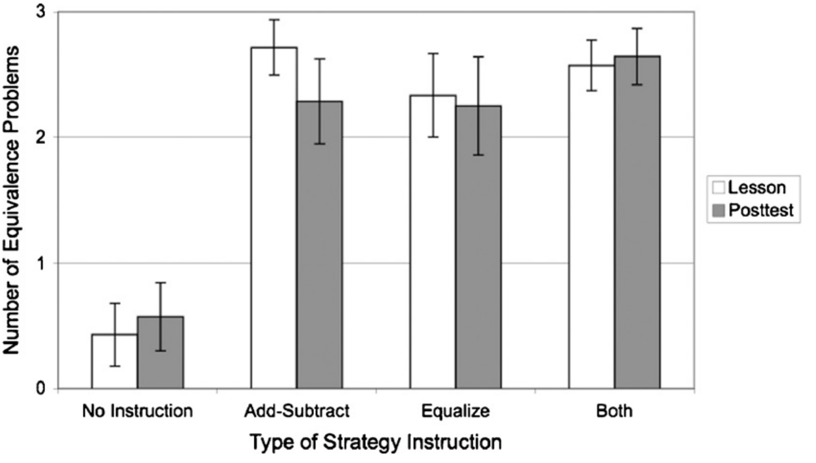

Fig. 3. Mean number of equivalence problems solved using correct strategies during the lesson and on the posttest (of three possible) for participants in each condition. The error bars represent standard errors.

variable. The results are presented in Fig. 3. There were significant main effects of both add–subtract instruction, $F(1,50) = 15.14$, $p < 0.001$, $\eta = 0.21$, and equalize instruction, $F(1,50) = 4.52$, $p = 0.04$, $\eta = 0.062$. The interaction was non-significant, $F(1,50) = 2.57$, $p = 0.12$, $\eta = 0.036$. Participants in all of the instructed groups used correct strategies at both lesson and posttest more frequently than participants in the no-instruction group. There was no effect of time point (lesson vs. posttest), and there were also no interactions involving time point.

Did participants adopt the taught strategies? By and large, they did. Table 4 presents the proportion of participants who used each of the most common strategies at pretest, lesson, and posttest. In the add–subtract condition, 79% of participants used the add–subtract strategy at posttest. In the equalize condition, 67% of participants used the equalize strategy at posttest. In the both-equalize-and-add–subtract condition, 43% of participants used both taught strategies at posttest, 29% used add–subtract only, and 21% used equalize only. Overall, 80% of participants in the experimental conditions who used incorrect strategies at pretest adopted a taught strategy. A few children in the control

Table 4
Mean number of trials on which each strategy was used, and percent of children who used each strategy at least once (in parentheses) at pretest, lesson, and posttest for children in each condition.

		No instruction	Add–subtract	Equalize	Both
Add all	Pretest	2.79 (100)	2.50 (93)	2.58 (92)	2.79 (100)
	Lesson	2.43 (93)	0.00 (0)	0.25 (8)	0.14 (7)
	Posttest	2.29 (93)	0.71 (29)	0.33 (17)	0.21 (14)
Add to equal sign	Pretest	0.07 (7)	0.21 (21)	0.33 (17)	0.00 (0)
	Lesson	0.00 (0)	0.07 (7)	0.33 (17)	0.07 (7)
	Posttest	0.00 (0)	0.00 (0)	0.42 (17)	0.07 (7)
Add–subtract	Pretest	0.00 (0)	0.00 (0)	0.00 (0)	0.00 (0)
	Lesson	0.07 (7)	2.71 (93)	0.00 (0)	1.36 (71)
	Posttest	0.21 (14)	2.29 (79)	0.00 (0)	1.50 (71)
Equalize	Pretest	0.00 (0)	0.00 (0)	0.00 (0)	0.00 (0)
	Lesson	0.29 (21)	0.00 (0)	2.25 (83)	1.21 (71)
	Posttest	0.21 (21)	0.00 (0)	2.00 (67)	1.14 (64)
Group	Pretest	0.00 (0)	0.00 (0)	0.00 (0)	0.00 (0)
	Lesson	0.07 (7)	0.00 (0)	0.17 (8)	0.00 (0)
	Posttest	0.14 (7)	0.00 (0)	0.50 (17)	0.00 (0)

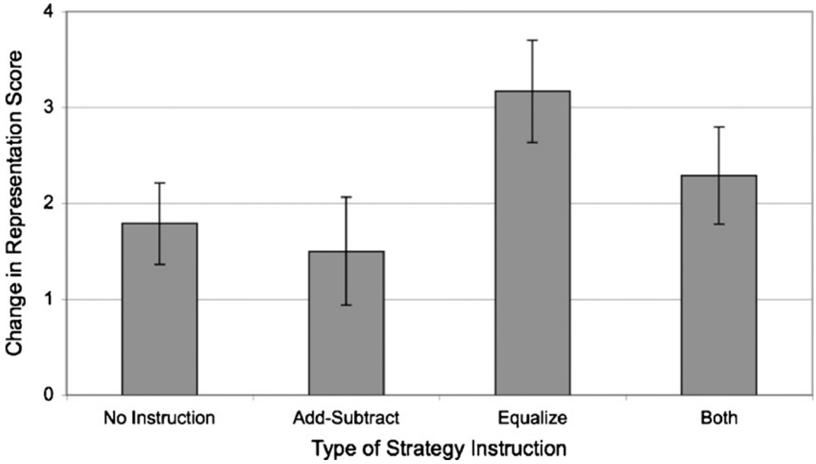

Fig. 4. . Mean pretest-to-posttest change in representation score for participants in each condition. The error bars represent standard errors.

condition also generated one or both of the taught strategies, despite not receiving instruction on either of them: 14% of participants in the control condition generated both add–subtract and equalize strategies, and 7% of participants in the control condition generated the equalize strategy only.

To assess whether participants acquired the equalize strategy, we conducted a 2 (equalize instruction: yes or no) × 2 (time point: lesson or posttest) ANOVA, with number of trials on which the equalize strategy was used as the dependent variable. As expected, participants who received equalize instruction used the equalize strategy much more frequently than participants who did not, $F(1,52) = 42.70$, $p < 0.0001$, $\eta = .45$. Similarly, to assess whether participants acquired the add–subtract strategy, we conducted a 2 (add–subtract instruction: yes or no) × 2 (time point: lesson or posttest) ANOVA, with number of trials on which the add–subtract strategy was used as the dependent variable. As expected, participants who received add–subtract instruction used the add–subtract strategy much more frequently than participants who did not, $F(1,52) = 78.50$, $p < 0.0001$, $\eta = 0.60$. The effect of time point and the interaction of instruction condition with time point were not significant in either analysis.

Thus, participants in the experimental conditions did, indeed, adopt the taught strategies, leaving us in a position to address the main question of interest—whether participants improved their problem representations as a function of learning new strategies. We addressed this question by examining performance on the representation pretest and posttest as a function of strategy instruction condition.

2.3. Problem representation

To evaluate whether learning new strategies led to changes in problem representation, we used improvement in representation (i.e., difference between posttest and pretest representation scores) as the dependent variable. A 2 (equalize instruction: yes or no) × 2 add–subtract instruction: yes or no) ANOVA revealed a significant main effect of equalize instruction, $F(1,50) = 4.52$, $p = 0.04$, $\eta = 0.08$. As seen in Fig. 4, participants who received equalize instruction improved their problem representations more from pretest to posttest than participants who did not. The effect of add–subtract instruction and the interaction were not significant. Thus, instruction about the equalize strategy led to improvements in problem representation, but instruction about the add–subtract strategy did not. Moreover, instruction about two problem-solving strategies (both equalize and add–subtract) was not more effective at promoting correct problem representation than instruction about a single strategy.

Participants in the no-instruction condition also displayed substantial improvement in their problem representations. Those participants in the no-instruction condition who spontaneously generated correct strategies showed greater improvements in problem representation than those who did not generate correct strategies ($M = 3.00$, SE = 0.70, vs. $M = 1.30$, SE = 0.45, $t(12) = 2.03$, $p < 0.03$, one-tailed, $d = 1.20$).

2.4. Transfer

Finally, we examined performance on the transfer test. We expected that participants who received instruction about the equalize strategy would show strong transfer performance, because the equalize strategy generalizes fairly straightforwardly to all problem types included in the transfer test. We further expected that participants who received instruction about the add–subtract strategy would show relatively weak transfer performance, because the add–subtract strategy must be adapted fairly substantially for problems that involve multiplication (four of the eight problems on the transfer test). Indeed, some participants who received instruction about the add–subtract strategy used an incorrect generalization of the add–subtract strategy on the multiplication transfer problems—namely, they multiplied the numbers on the left side and then subtracted the number on the right. In addition to examining effects of the instruction conditions on transfer performance, we also wished to examine whether problem representation contributed to transfer performance.

We analyzed the full dataset ($N = 89$) using multiple regression, with pretest representation score and number of correct strategies used at pretest as continuous predictor variables, and with equalize instruction (coded 0 or 1) and add–subtract instruction (coded 0 or 1) as categorical predictor variables. Because the multiplication problems require a more generalized understanding of equivalence than the addition transfer problems, we analyzed performance on the addition and multiplication transfer problems separately.

Scores on the addition transfer problems were significantly predicted by correct strategies at pretest, $\beta = 0.50$, $t(84) = 5.13$, $p < 0.001$ and by equalize instruction, $\beta = 0.19$, $t(84) = 2.16$, $p = 0.03$, but not by add–subtract instruction or by pretest representation scores. The regression model explained a significant proportion of the variance in addition transfer scores, $R^2 = 0.36$, $F(4,84) = 11.99$, $p < 0.001$.

Scores on the multiplication transfer problems were significantly predicted by correct strategies at pretest, $\beta = 0.58$, $t(84) = 6.76$, $p < 0.0001$, by equalize instruction, $\beta = 0.17$, $t(84) = 2.23$, $p = 0.03$, and by pretest representation scores, $\beta = 0.20$, $t(84) = 2.26$, $p = 0.03$, but not by add–subtract instruction. The regression model explained a significant proportion of the variance in multiplication transfer scores, $R^2 = 0.51$, $F(4,84) = 21.85$, $p < 0.001$.

Thus, add–subtract instruction did not significantly influence transfer performance, but equalize instruction contributed to successful transfer to both addition and multiplication problems. For addition problems, participants who received equalize instruction solved an average of 1.98 (of three) problems correctly, whereas those who did not receive equalize instruction solved an average of 1.51 problems correctly. For multiplication problems, participants who received equalize instruction solved an average of 2.13 (of four) problems correctly, whereas those who did not receive equalize instruction solved an average of 1.56 problems correctly. Finally, children who represented the problems more accurately at pretest also displayed better transfer to multiplication problems.

3. Discussion

In this study, children who received a lesson about a new problem-solving strategy improved their problem representations. Thus, direct instruction about strategies can lead to changes in problem representation. However, this held true only for children who were taught the equalize strategy, not for those who were taught the add–subtract strategy. Thus, some strategies appear to be "better" at promoting accurate problem representation than others. It is important to note that neither type of strategy instruction included a conceptual rationale—so it was instruction about the equalize strategy itself, and not about the rationale for the equalize strategy, that led to improvements in representation.

Why might one strategy but not the other lead to improvements in representation? One possible answer lies in the relation between the actions involved in applying the strategy and the learner's prior problem representation. The actions involved in the equalize strategy focus on both *sides* of the equation. Research has shown that many elementary school children fail to understand the term "side" in this context (Rittle-Johnson & Alibali, 1999), and moreover, many children make errors in representing the right side of mathematical equivalence problems (McNeil & Alibali, 2004). Thus, the focus on the right side of the equation in the equalize instruction may have promoted accurate representation.

In contrast, the add–subtract strategy focuses on the *number* on the right side of the equation, rather than on the right *side* of the equation *per se*. Children encoded the numbers in the problem well even before instruction; indeed, children made many fewer number errors than conceptual errors on the pretest reconstruction task (there were 54 number errors vs. 104 conceptual errors made by children who used incorrect strategies at pretest). It appears that the add–subtract instruction highlighted a feature of the problems that children already encoded accurately (the number on the right side), and therefore it did little to improve children's encoding of the problems. Instead, the add–subtract instruction simply encouraged children to use a different operation (subtraction rather than addition) on an already-encoded feature.

Instruction about the equalize strategy also promoted successful transfer to both addition and multiplication problems, whereas instruction about the add–subtract strategy did not. As expected, the equalize strategy could be straightforwardly generalized to both addition and multiplication problems. It is not clear why add–subtract instruction did not promote transfer to addition problems, as it would seem that the add–subtract strategy could be straightforwardly generalized to such problems. However, it is less surprising that add–subtract instruction did not promote transfer to multiplication problems, because to be transferred successfully, it would need to be adapted to "multiply–divide". This is a substantial adaptation, and children apparently did not have the requisite knowledge about inverse operations to make this adaptation successfully.

Accurate problem representation was also important for transfer to multiplication problems. Children who represented the problems accurately at pretest were better at adapting their successful problem-solving strategies to multiplication transfer problems than were children who represented the problems poorly. Accurate problem representation may guide correct adaptation of strategies to novel problem types, or it may serve to prevent faulty adaptations.

However, all transfer problems were relatively "near" transfer. It is possible that even accurate problem representation and equalize instruction would have been ineffective at promoting transfer if we had used "far" transfer problems that require other elements of knowledge for successful solution (e.g., problems with the unknown on the left side of the problem, problems with variables).

This study has shown that learning a new strategy can lead to improvements in problem representation. However, learning a new strategy is certainly not the only source of improved problem representation (Alibali, 2005). Indeed, we found that children in the no-instruction condition improved their problem representations from pretest to posttest, even though they received no instruction of any kind. This held true both for children in the no-instruction condition who spontaneously invented correct procedures, and for those who did not (although improvements were greater among those who invented correct procedures). It is possible that this finding could be due to measurement error and regression to the mean; however, this finding is also compatible with the idea that problem representation improves as a function of mere exposure to problems, via mechanisms of implicit learning (Alibali, Ockuly, & Fischer, 2005), and it highlights the importance of understanding other sources of change in problem representation besides learning new strategies. Other sources of change in representation may be at least as powerful as learning new strategies. For example, conceptually oriented instruction (e.g., a lesson about the principle that the two sides of an equation represent the same quantity) would likely also evoke substantial improvements in problem representation.

This study focused on improvements in children's representations of equivalence problems. Problem representation is only one aspect of children's knowledge in this task domain. Children may also have procedural knowledge about how to solve the problems, and they may also have conceptual knowledge about equations, such as understanding the idea that the two sides of an equation need to be equivalent and knowing that the equal sign symbolizes the equivalence relation. These aspects of knowledge are linked (Rittle-Johnson & Siegler, 1999), but not identical. Indeed, it is possible that children could represent the equivalence problems perfectly, but have inadequate procedural or conceptual knowledge to correctly solve problems or transfer procedures to novel equations. For example, a child might represent the presence and position of the equal sign in the problems but fail to understand that the equal sign means "the same as". Such a child might perform perfectly on assessments of problem representation (such as the reconstruction and recognition tasks used in this study) but be unable to generate a correct strategy for solving the problems. A few children in our sample ($N = 4$) fit this profile, in that they performed at ceiling on the representation pretest but still solved the pretest problems incorrectly. For these children, learning a new strategy could not improve their already-accurate problem representations but would likely lead to changes in their procedural knowledge.

Some other limitations of the present study must also be acknowledged. First, our analyses of changes in problem representation were based on just over half of the original sample, and those who were included were the less able math students (i.e., those who did not succeed on the equivalence problems at pretest). Had we tested younger, more able students, our findings may have been different. Second, the implications of the present findings for educational practice are limited, because the instruction that we provided in our experimental manipulation was not representative of instruction that would typically be given in classrooms. Despite these limitations, however, the present findings are of value because they provide the first causal evidence that instruction about new strategies leads to changes in problem representation.

Understanding learning and development in domains that involve problem solving will require understanding possible sources of change in problem representation. The present findings indicate that learning new strategies is one source of such change. However, they also suggest that some new strategies lead to improvements in representation more effectively than others. The key seems to be the relation between the requirements of the new strategy and the nature of the gaps in children's initial problem representations. More generally, the present findings converge with past research to build the case that the relation between problem representation and strategy use is a reciprocal one. Past research has shown that improvements in problem representation can lead to changes in strategy use, and the present study demonstrates that learning new strategies can lead to improvements in problem representation.

Appendix A

Problems on the transfer test.

1. $4 + 7 + 9 = 4 + \_\_$
2. $8 + 4 + 3 = \_\_ + 3$
3. $6 + 9 + 4 = 5 + \_\_$
4. $7 + 3 + 8 = \_\_ + 6$
5. $5 \times 3 \times 2 = 5 \times \_\_$
6. $4 \times 5 \times 3 = \_\_ \times 3$
7. $3 \times 5 \times 2 = 6 \times \_\_$
8. $2 \times 4 \times 3 = \_\_ \times 6$

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